HETEROGENEOUS AGENT MODELS IN ECONOMICS AND FINANCE*

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Contents

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Abstract.

This chapter surveys work on dynamic heterogeneous agent models (HAMs) in economics and finance. Emphasis is given to simple models that, at least to some extent, are tractable by analytic methods in combination with computational tools. Most of these models are behavioral models with boundedly rational agents using different heuristics or rule of thumb strategies that may not be perfect, but perform reasonably well. Typically these models are highly nonlinear, e.g. due to evolutionary switching between strategies, and exhibit a wide range of dynamical behavior ranging from a unique stable steady state to complex, chaotic dynamics. Aggregation of simple interactions at the micro level may generate sophisticated structure at the macro level. Simple HAMs can explain important observed stylized facts in financial time series, such as excess volatility, high trading volume, temporary bubbles and trend following, sudden crashes and mean reversion, clustered volatility and fat tails in the returns distribution.

Keywords:

interacting agents, behavioral economics, evolutionary finance, complex adaptive systems, nonlinear dynamics, numerical simulation.

JEL classification: B4, C0, C6, D84, E3, G1, G12

"*One of the things that microeconomics teaches you is that individuals are not alike. There is heterogeneity, and probably the most important heterogeneity here is heterogeneity of expectations. If we didn't have heterogeneity, there would be no trade. But developing an analytic model with heterogeneous agents is difficult.*" (Ken Arrow, In: D. Colander, R.P.F. Holt and J. Barkley Rosser (eds.), *The Changing Face of Economics. Conversations with Cutting Edge Economists.* The University of Michigan Press, Ann Arbor, 2004, p301.)

1 Introduction

Economics and finance are witnessing an important paradigm shift, from a representative, rational agent approach towards a behavioral, agent-based approach in which markets are populated with boundedly rational, heterogeneous agents using rule of thumb strategies. In the traditional approach, simple analytically tractable models with a representative, perfectly rational agent have been the main corner stones and mathematics has been the main tool of analysis. The new behavioral approach fits much better with agent-based simulation models and computational and numerical methods have become an important tool of analysis. In the recent literature however, already quite a number of heterogeneous agent models (HAM) have been developed which, at least to some extent, are analytically tractable and for which theoretical results have been obtained supporting numerical simulation results. In this chapter we review a number of *dynamic* HAM in economics and finance. Most of these models are concerned with financial market applications, but some of them deal with different markets, such as commodity good markets. The models reviewed in this chapter may be viewed as simple, stylized versions of the more complicated "artifical markets" and computationally oriented agent-based simulation models reviewed in the chapter of LeBaron (2005) in this handbook. In the analysis of the dynamic HAM discussed in the current chapter one typically uses a mixture of analytic and computational tools.

The new behavioral, heterogeneous agents approach challenges the traditional representative, rational agent framework. It is remarkable however, that many ideas in the behavioral, agent-based approach in fact have quite a long history in economics already dating back to earlier ideas well before the rational expectations and efficient market hypotheses. For example, some of the key elements of the behavioral agent-based models are closely related to Keynes' view that '*expectations matter*', to Simon's view that economic man is *boundedly rational* and to the view of Kahneman and Tversky in psychology that individual behavior under uncertainty can best be described by simple *heuristics and biases*. Before starting our survey, we briefly discuss these important (and closely related) ideas, which will be recurrent themes in this chapter.

Keynes (1936) argued that investors' sentiment and market psychology play an important role in financial markets, as will be clear from the following famous quote: '*Investment based on genuine long-term expectation is so difficult as to be scarcely practicable. He who attempts it must surely lead much more laborious days and run greater risks than he who tries to guess better than the crowd how the crowd will behave; and, given equal intelligence, he may make more disastrous mistakes*' (Keynes, 1936, p.157). According to Keynes, it is hard to compute an objective measure of 'market fundamentals' and, if possible at all, it is costly to gather all relevant information. Another difficulty is that it is not clear what the 'correct' fundamental variables are, and fundamentals can be relevant only when enough traders agree on their role in determining asset prices. Instead of relying on market fundamentals, for an investor it may be easier, less risky and more relevant to make a rule of thumb estimate of the market sentiment. Herbert Simon (1957) emphasized that individuals are limited in their knowledge about their environment and in their computing abilities, and moreover that they face search costs to obtain sophisticated information in order to pursue optimal decision rules. Simon argued that, because of these limitations, *bounded rationality* with agents using simple but reasonable or satisficing rules of thumb for their decisions under uncertainty, is a more accurate and more realistic description of human behavior than perfect rationality with fully optimal decision rules. In the seventies this view was supported by evidence from psychology laboratory experiments of Kahneman and Tversky (1973) and Tversky and Kahneman (1974), showing that in simple decision problems under uncertainty humans do not behave rational, in the sense of maximizing expected utility, but their behavior can be described by *simple heuristics* which may lead to significant *biases*. For a more recent and stimulating discussion of bounded rationality, simple heuristics and biases as opposed to rational behavior we refer to the Nobel Memorial Lectures in Simon (1979) and Kahneman (2003).

In contrast, Milton Friedman has been one of the strongest advocates of a rational agent approach, claiming that the behavior of consumers, firms and investors can be described *as if* they behave rationally. The *Friedman hypothesis* stating that non-rational agents will not survive evolutionary competition and will therefore be driven out of the market has played an important role in this discussion. The following quote from Friedman (1953, p.175) concerning non-rational speculators is well known: *'People who argue that speculation is generally destabilizing seldom realize that this is largely equivalent to saying that speculators* lose *money, since speculation can be destabilizing in general only if speculators on the average sell when the currency is low in price and buy when it is high'*. In a similar spirit, Alchian (1950) argued that biological evolution and natural selection driven by realized profits may eliminate non-rational, non-optimizing firms and lead to a market where rational, profit maximizing firms dominate. The question whether the Friedman hypothesis holds in a heterogeneous world has played an important role in the development and discussion about HAMs, and we will come back to it several times in this chapter.

Rational behavior has two related but different aspects (e.g. Sargent (1993)). Firstly, a rational decision rule has some micro-economic foundation and is derived from *optimization principles*, such as expected utility or expected profit maximization. Secondly, agents have *rational expectations* (RE) about future events, that is, beliefs are perfectly consistent with realizations and a rational agent does not make systematic forecasting errors. In a rational expectations equilibrium, forecasts of future variables coincide with the mathematical conditional expectations, given all relevant information. Rational expectations provided an elegant and parsimonious way to exclude 'ad hoc' forecasting rules and market psychology from economic modeling. Since its introduction in the sixties by Muth (1961) and its popularization in economics by Lucas (1971), the rational expectations hypothesis (REH) has become the dominating expectation formation paradigm in economics.

Another important issue in the discussion of rational versus boundedly rational behavior is concerned with *market efficiency*, as e.g. emphasized by Fama (1965). If markets were not efficient, then there would be unexploited profit opportunities, that would be exploited by rational arbitrage traders. Rational traders would buy (sell) an underpriced (overpriced) asset, thus driving its price back to the correct, fundamental value. In an efficient market, there can be *no forecastable structure* in asset returns, since any such structure would be exploited by rational arbitrageurs and therefore disappear.

In the seventies and eighties the representative agent, rational expectations and efficient market hypotheses became the dominating paradigm in economics and finance. In the late eighties and nineties however, HAMs and bounded rationality became increasingly popular. The following developments contributed to this change:

1. In a world where all agents are rational and it is common knowledge that all agents are rational, there will be *no trade*. A trader with superior private information can not benefit from his information, because other rational traders anticipate that he must e.g. have positive information about an asset and will therefore not sell the asset to him. Several *no trade theorems* have been obtained (Milgrom and Stokey (1982); see Fudenberg and Tirole (1991, especially section 14.3.3) for a discussion). No trade theorems are in sharp contrast with the high daily trading volume observed in real markets, such as the stock market and the foreign exchange market. This tremendous trading volume reinforces the idea of heterogeneous expectations and the idea that it takes differences of opinion among market participants to trade.

2. In the early eighties, Shiller (1981) and LeRoy and Porter (1981) claimed that stock prices exhibit *excess volatility*, that is, movements in stock prices are much larger than movements in underlying economic fundamentals. Statistical tests for excess volatility were developed, but the power of these tests turned out to be low and the issue is still heavily debated. The stock market crash in October 1987 reinforced the idea of excess volatility and the crash appeared to be difficult to explain by a representative, rational agent model. Another important empirical observation has been the strong appreciation followed by a strong depreciation of the dollar in the mid eighties, which seemed to be unrelated to economic fundamentals as stressed by Frankel and Froot (1986). Cutler, Poterba and Summers (1989) showed that the days of the largest aggregate stock market movements in the S&P500 index, 1941-1987, do not coincide with the days of the most important fundamental news and vice versa. These empirical observations have played an important role in the increasing popularity of non-rational, heterogeneous agent explanations of asset price movements.

3. Following earlier ideas of Simon, in the nineties and since more and more economists have come to question the unrealistically strong rationality assumptions concerning perfect information about the environment and unlimited computing abilities. In particular, in a *heterogeneous* world a rational agent has to know the beliefs of *all* other, *non-rational* agents, which seems highly unrealistic as emphasized e.g. in Arthur (1995) and Hommes (2001). These developments contributed to a rapidly growing interest in *bounded rationality* in the 1990s, see for example the survey by Sargent (1993). A boundedly rational agent forms expectations based upon *observable* quantities and adapts his forecasting rule as additional observations become available. *Adaptive learning* may converge to a rational expectations equilibrium or it may converge to an "approximate rational expectations equilibrium", where there is at least some degree of consistency between expectations and realizations (see Evans and Honkapohja (2001) for an extensive and modern treatment of adaptive learning in macroeco-

nomics).

4. A problem with behavioral economics and bounded rationality is that it leaves "*many degrees of freedom*". Any HAM with bounded rationality must provide a plausible story that there is at least some reasonable consistency between beliefs and realizations and how agents select from a large class of possible forecasting and trading strategies. One plausible story is an *evolutionary approach*, advocated by Nelson and Winter (1973,1974,1982), where agents or firms select from a class of simple, behavioral strategies according to their relative performance, e.g. as measured by relative profitability and how much this strategy is used by others. The evolutionary approach plays an important role in this chapter.

5. New developments in mathematics, physics and computer science in *nonlinear dynamics*, *chaos* and *complex systems* motivated economists to apply these tools. Economic applications of nonlinear dynamics are surveyed in Brock et al. (1991), Day (1994), Lorenz (1993) and Medio (1992). The Santa Fe conference proceedings Anderson et al. (1988) and Arthur et al. (1997) contain contributions in which the economy is viewed as a *complex evolving system*, see also Arthur (2005) and the collection of papers in Rosser (2004a). Nonlinear dynamics, chaos, and complex systems have important consequences for the validity of the REH. In a simple (linear) stable economy with a unique steady state path, it seems natural that agents can learn to have rational expectations, at least in the long run. A representative, perfectly rational agent model nicely fits into a linear view of a globally stable and predictable economy. But how could agents have rational expectations or perfect foresight in a complex, nonlinear world, with prices and quantities moving irregularly on a strange attractor? A boundedly rational world view with agents using simple forecasting strategies, perhaps not perfect but at least approximately right, seems more appropriate within a complex, nonlinear world; see e.g. Brock and Hommes (1997b). Applications of tools from nonlinear dynamics and complex systems theory have stimulated much work in HAM, which are almost always highly nonlinear, adaptive systems.

6. Laboratory experiments have shown that individuals often do not behave rationally. We already mentioned the work by Kahneman and Tversky, showing that individuals tend to use heuristics and biases in making decisions under uncertainty. In a stimulating and influential paper, Smith et al. (1988) showed the occurrence of bubbles in *asset pricing laboratory experiments*; see also the survey in Sunder (1995). These bubbles occur despite the fact that participants had sufficient information to compute the fundamental value of the asset. This type of laboratory experiments reinforced theoretical work on HAMs with non-rational agents. See also the chapter of Duffy (2005) on the relationship between laboratory experiments and agent-based modeling.

7. Evidence from *survey data* on exchange rate expectations of financial specialists, e.g. by Frankel and Froot (1987ab, 1990ab), Allen and Taylor (1990), Ito (1990) and Taylor and Allen (1992), showed that financial practitioners use different trading and forecasting strategies. A consistent finding from survey data is that at short horizons investors tend to use extrapolative chartists' trading rules, whereas at longer horizons investors tend to use mean reverting fundamentalists' trading rules. Frankel and Froot (1987b, p.264) conclude the following from their survey data analysis: *"It may be that each*

respondent is thinking to himself or herself, "I know that in the long run the exchange rate must return to the equilibrium level dictated by fundamentals. But in the short run I will ride the current trend a little longer. I only have to be careful to watch for the turning point and to get out of the market before everyone else does"." For a long time academic work has been skeptical concerning the usefulness of technical trading. Brock et al. (1992) tested 26 simple, frequently used technical trading rules (e.g. moving average and trading range breaks) on the Dow Jones index in the period 1897-1986 and showed that they can generate significantly positive returns, suggesting extra structure above and beyond the EMH benchmark. Dacorogna et al. (1995) show that trading models with different time horizons and risk profiles can be profitable when applied to high frequence exchange rate data. Both the work on survey data, the fact that technical trading is used extensively among practitioners and empirical work suggesting the potential success of technical trading have stimulated much work on HAMs with chartists versus fundamentalists.

8. Finally, the fact that computational tools became widely available in the late eighties and the nineties has enormously stimulated the development and numerical simulation analysis of behavioral HAMs with boundedly rational agents, both in research and in teaching. The current Handbook provides the best proof of this fact, see in particular the chapters of Judd (2005) and Tesfatsion (2005).

There is already too much work on HAMs to provide a comprehensive review in this chapter. We focus on stylized dynamic HAMs using some simple examples to illustrate and discuss what we believe to be important characteristics of HAMs. A long list of references is provided to help guide the interested reader through the already extensive literature. The chapter of LeBaron (2005) contains an overview of larger, computational HAMs as well as many more references to the literature. This chapter is organized as follows. Section 2 discusses some early HAMs with chartists and fundamentalists and work on survey data analysis of expectations of financial experts. Section 3 relates the work on HAMs to behavioral finance. Section 4 presents examples of disequilibrium HAMs, where the interaction of agents leads to complex market dynamics such as cycles or chaotic fluctuations. Section 5 discusses stochastic interacting agent systems and work on social interactions. Section 6 discusses simple financial market HAMs with herding behavior, able to generate important stylized facts such as clustered volatility. Section 7 discusses models where sophisticated agents using advanced but costly strategies compete against simple agents using cheap rule of thumb strategies. Section 8 discusses an asset pricing model with heterogeneous beliefs with endogenous evolutionary switching of strategies. Section 9 summarizes and discusses some future perspectives.

2 Fundamentalists and chartists

In many HAMs two important types of agents are distinguished, *fundamentalists* and *chartists*. Fundamentalists base their expectations about future asset prices and their trading strategies upon market fundamentals and economic factors, such as dividends, earnings, macroeconomic growth, unemployment rates, etc. They tend to invest in assets that are undervalued, that is, whose prices are below a benchmark fundamental value, and sell assets that are overvalued, that is, whose prices are above the market fundamental value. In contrast, chartists or technical analysts do not take market fundamentals into account but instead base their expectations about future asset prices and their trading strategies upon observed historical patterns in past prices. Technical analysts try to extrapolate observed price patterns, such as trends, and exploit these patterns in their investment decisions. A well known example of a technical trading rule is the *moving average* trading rule, buying (selling) an asset when a short run moving average crosses a long run moving average from below (above).

This section discusses some early work emphasizing the importance of fundamentalists and chartists. Subsection 2.1 discusses one of the first financial HAMs with fundamentalists and chartists, due to Zeeman (1974). Subsection 2.2 discusses work on survey data on expectations of Frankel and Froot (1986,1987ab,1990ab), Allen and Taylor (1990) and Taylor and Allen (1990), showing the importance of chartists trading rules among financial practitioners. Finally, subsection 2.3 discusses another early model with fundamentalists and chartists discussed in a series of papers by Frankel and Froot (1986,1987ab,1990ab), which have stimulated much subsequent work in this area.

2.1 An early example

One of the first HAMs for the stock market (or for exchange rates) can be found in Zeeman (1974). This model is an application of the *cusp catastrophe with a slow feedback flow*. Zeeman's purpose was to offer a qualitative description of the observed stylized fact of temporary bull and bear markets. The model is very stylized and lacks any micro foundations, but nevertheless it contains a number of important, behavioral elements that have also been used in recent heterogeneous agents modeling.

The model contains two types of traders, fundamentalists and chartists. Fundamentalists know the 'true' value of the stock and buy (sell) when the price is below (above) that value. Chartists are trend followers, buying when price rises and selling when price falls. There are three variables J, F and C. J denotes the rate of change of a stock market index or of an exchange rate. $J = 0$ represents a static market, whereas $J > 0$ ($J < 0$) represents a bull (bear) market. C denotes the proportion of the market held by chartists, i.e. the proportion of speculative money in the market, and F denotes the excess demand for stock by fundamentalists. Zeeman assumes that J responds to C and F much faster than C and F respond to J. Stated differently, J is a fast variable (a state variable) and C and F are slow variables (control variables or slowly changing parameters).

Zeeman postulates seven hypotheses based upon observed qualitative features of the stock exchange and the behavior of speculators (chartists) and value investors (fundamentalists). Using Thom's classification theorem, Zeeman then shows that the simplest generic mathe-

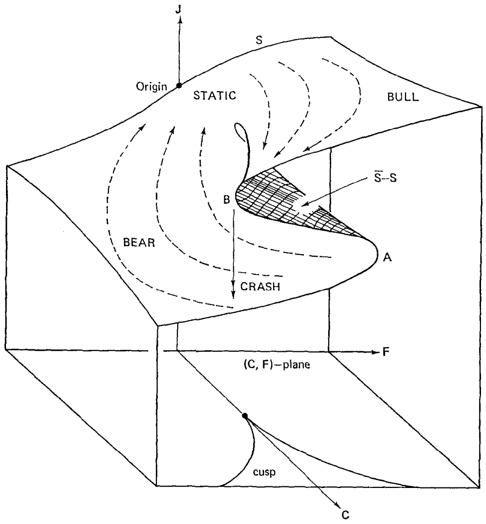

Figure 1: The cusp catastrophe surface of equilibria. The state variable J is the rate of change of the stock index, whereas the control variables F and C represent excess demand of fundamentalists and the proportion of chartists. For clarity the (C, F)-plane has not been drawn through the origin, but below the surface. Reprinted with permission from *Journal of Mathematical Economics*, Vol. 1, No. 1, 1974, E.C. Zeeman, The unstable behavior of stock exchange, Figure 3, p. 46.

matical model that can be derived from these hypotheses is the cusp catastrophe model with a slow feedback flow. Figure 1 shows the surface S satisfying

$$J^3 - (C - C_0) - F = 0. \tag{1}$$

The surface S represents the equilibria of the system[1]. The projection of S onto the $(C, F)-$plane yields the cusp region, bounded by two fold curves tangent to each other in the cusp point. For (C, F) outside the cusp region, S is single sheeted and the model has a unique (stable) steady state. Inside the cusp region, S is 3-sheeted, the middle sheet representing an unstable equilibrium and the other two sheets stable equilibria. The system converges quickly to the attractor surface S and then slowly moves along the surface. For example, consider a situation where the system is in a bull market at the upper sheet of S. In a bull market the proportion of chartists increases, because they 'follow the trend', thus accelerating a further increase of the stock index. At some point however fundamentalists start selling stocks, because they judge that the market has become overvalued, causing the growth of the index to decrease. The excess demand F of fundamentalists decreases and the system moves along the upper sheet of S in the direction of the point B, causing a crash and a rapid decline of the stock prices until the system reaches the lower sheet of S. During this bear market, at some point fundamentalists start buying stocks again, because they believe that the stock is undervalued, causing a decrease in the proportion of chartsits and an increase of the market index. As the index rises, the proportion of chartists increases again, accelerating the rise in stock prices leading to a new bull market. Zeeman's model thus explains a switching between bull and bear markets, as indicated by the arrows in Figure 1, derived from behavioral assumptions about chartists and fundamentalists.

[1]Notice that, since J denotes the rate of change, these equilibria are not steady states, but rather equilibria with constant growth rate.

Catastrophe theory became quite popular in the early seventies, but was heavily criticized as being a “science fad” in the late seventies and eighties, for example in Zahler and Sussmann (1977). Rosser (2004b) contains an interesting recent discussion and reappraisal of mathematical methods from catastrophe theory and of Zeeman’s model. Guastello (1995, pp.292-297) studied the 1987 market crash using Zeeman’s model, whereas recently Rheinlaender and Steinkamp (2004) introduced a stochastic version of Zeeman’s model using random dynamical systems theory.

2.2 Survey data on expectations

In the late eighties and early nineties, a number of authors including Frankel and Froot (1987, 1990), Shiller (1987), Allen and Taylor (1990) and Taylor and Allen (1992) conducted questionnaires among financial practitioners to obtain detailed information about investors’ expectations. This survey data work has been an important source of inspiration for the development of HAMs. More recent survey based evidence includes Cheung et al. (1999), Lui and Mole (1998) and Menkhoff (1997,1998).

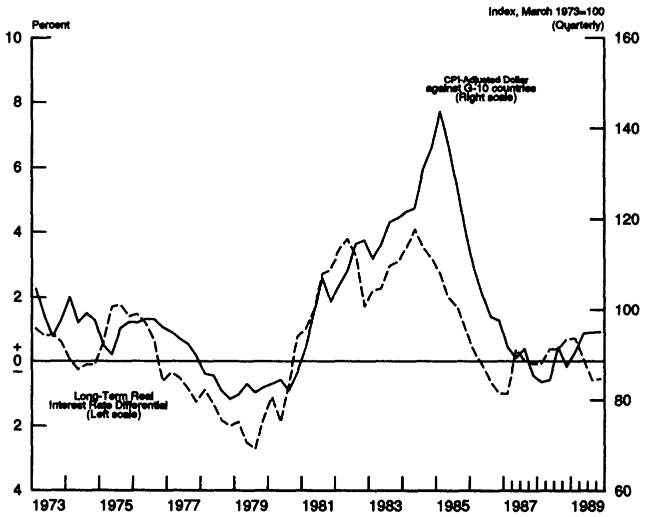

Figure 2: Time series of the real value of the dollar against a weighted average of the currencies of the foreign G-10 countries plus Switzerland (bold graph) and the time series of the real interest differential between the US and a weighted average of the foreign country rates (dotted graph) in the eighties. Reprinted with permission from *American Economic Review*, Vol. 80, No. 2, AEA Papers and Proceedings, Frankel, J.A. and Froot, K.A., The rationality of the foreign exchange rate. Chartists, fundamentalists and trading in the foreign exchange market, Figure 1, p.181.

In their series of papers in the mid eighties and early nineties, Frankel and Froot studied the large movements of the US dollar exchange rate in the eighties and in particular they investigated the question whether investors’ expectations may have amplified those movements. Frankel and Froot (1990a) contains a detailed description of this research; a short, but stimulating discussion is also given in Frankel and Froot (1990b). Figure 2 shows a time series of the real value of the dollar against a weighted average of the currencies of the foreign G-10 countries plus Switzerland and the time series of the real interest differential between the US and a weighted average of the foreign country rates in the eighties. Frankel and Froot

(1990b, p.181) note the following:

"At times, the path of the dollar has departed from what would be expected on the basis of macroeconomic fundamentals. The most dramatic episode is the period from June 1984 to February 1985. The dollar appreciated another 20 percent over this interval, even though the real interest differential had already begun to fall. The other observable factors that are suggested in standard macroeconomic models (money growth rates, real growth rates, the trade deficit) at this time were also moving in the wrong direction to explain the dollar rise."

Indeed it seems difficult to believe that a rational theory could explain such an increase of 20% of the equilibrium real exchange rate within 9 months and that this rapid rise would then be reversed within the next month. Instead Frankel and Froot (1990b, p.182) argue that *"the appreciation may have been an example of a speculative bubble – that it was not determined by economic fundamentals, but rather was the outcome of self-confirming market expectations."* Frankel and Froot use survey data on exchange rate expectations to test this hypothesis[2].

Frankel and Froot (1987ab, 1990ab) use three different sources for their survey data on exchange rate expectations of financial specialists, bankers and currency traders. Some of the surveys go back to 1976 and include telephone interviews. The time horizon of the exchange rate expectations vary from 1 week to 12 months. An important finding is that respondents' short-term expectations are quite different from their long-term expectations. Frankel and Froot estimate three simple, standard models for expectations, namely extrapolative expectations, regressive (or mean reverting) expectations and adaptive expectations. The *extrapolative expectations* model assigns a weight g to the lagged spot rate and a weight $(1-g)$ to the current spot rate, that is, the expected spot rate is given by

$$s^e_{t+1} = (1-g)s_t + gs_{t-1}, \tag{2}$$

where s_t is the log of the current spot rate, or equivalently

$$\Delta s^e_{t+1} = -g\Delta s_t, \tag{3}$$

where Δs^e_{t+1} is the expected change of the (log) spot rate and Δs_t the last realized change. For short-term horizons (1 week, 2 weeks, 1 month) significantly negative values of g (ranging from -0.13 to -0.05) are obtained, characteristic of *destabilizing* or *bandwagon* expectations for which a current appreciation generates self-sustaining expectations of future appreciations. In contrast, at longer-term horizons of 6–12 months, significantly positive values of g (ranging from 0.07 to 0.38) are obtained characteristic of *stabilizing* expectations, where a trend is expected to reverse.

The *regressive* or *mean-reverting expectations* model is a weighted average between the current (log) spot rate and the (log) long-run equilibrium spot rate $\bar{s}_t$, that is

$$s^e_{t+1} = (1-v)s_t + v\bar{s}_t, \tag{4}$$

[2]Another rational explanation of the large fluctuations in the exchange rate is a time varying risk premium. Froot and Frankel (1989) show however that the bias in the forward discount, i.e. the log difference of the forward exchange rate and the spot rate, cannot be explained by rational expectations and a (time varying) risk premium, but may be attributable to systematic expectational errors.

or in terms of expected depreciation

$$\Delta s^e_{t+1} = v(\bar{s}_t - s_t). \tag{5}$$

If the weight v is positive (negative), then investors expect the exchange rate to move towards (away from) the long run equilibrium value. A negative weight v is characteristic of *destabilizing* or *explosive* expectations, while a positive weight v is characteristic of *stabilizing* expectations. Again, at short-term horizons (1 week, 2 weeks, 1 month) significantly negative values of v (ranging from -0.03 to -0.08) are obtained, whereas at longer-term horizons of 6–12 months significantly positive values of v (ranging from 0.06 to 0.17) are obtained. Similar results are also found for the case of adaptive expectations. Frankel and Froot (1990a, pp.98–101) conclude that

"... short-term and long-term expectation behave very differently from one another. In terms of the distinction between fundamentalists and chartists views, we associate the longer-term expectations, which are consistently stabilizing, with the fundamentalists, and the shorter term forecasts, which seem to have a destabilizing nature, with the chartists expectations. Within each of the above tables, it is as if there are actually two models of expectations operating, one at each end of the forecasting horizons, and a blend in between. Under this view, respondents use some weighted average of the chartist and fundamentalist forecasts in formulating their expectations for the value of the dollar at a given future date, with weights depending on how far off that date is."

This conclusion is in line with other questionnaire surveys of Allen and Taylor (1990) and Taylor and Allen (1992), conducted on behalf of the Bank of England, among chief foreign exchange dealers in London. Taylor and Allen (1992, p.304) conclude:

... at least 90 per cent of the respondents place some weight on this form of non-fundamental analysis when forming views at one or more time horizons. There is also a skew towards reliance on technical, as opposed to fundamentalist, analysis at shorter horizons, which becomes steadily reversed as the length of horizon considered is increased. A very high proportion of chief dealers view technical and fundamental analysis as complementary forms of analysis and a substantial proportion suggest that technical advice may be self-fulfilling.

Techniques used by Forecasting Services

Year	Total	Chart.	Fund.	Both
1978	23	3	19	0
1981	13	1	11	0
1983	11	8	1	1
1984	13	9	0	2
1985	24	15	5	3
1988	31	18	7	6

Table 1: From Frankel and Froot (1990b, p.184, Table 2); source: Euromoney, August issues.
Total = number of services surveyed; Chart. = number who reported using technical analysis; Fund. = number who reported using fundamentals models; and Both = number reporting a combination of the two. When a forecasting firm offers more than one service, each is counted separately.

Finally, Table 1 is reproduced from Frankel and Froot (1990b) showing how the relative importance of fundamentalist and technical analysis shifts over time. The table shows that in 1978 most of the forecasting services (19 vs. 3) relied on fundamental analysis, whereas in 1985 the situations has been reversed (5 vs. 15). Frankel and Froot (1990b, pp.184-185) conclude the following:

"... it may indeed be the case that shifts over time in the weight that is given to different forecasting techniques are a source of changes in the demand for dollars, and that large exchange rate movements may take place with little basis in macroeconomic fundamentals."

2.3 An exchange rate model

Their work on questionnaire surveys among financial practitioners motivated Frankel and Froot (1986,1990ab) to develop a heterogeneous agent model for exchange rates with time varying weights of forecasting strategies, which has stimulated much subsequent research in the field. Their exchange rate model contains three classes of agents: fundamentalists, chartists and portfolio managers. Fundamentalists think of the exchange rate according to a model –e.g. the overshooting model– that would be exactly correct if there were no chartists in the market. Chartists do not have fundamentals in their information set; instead they use autoregressive time series models –e.g. simple extrapolation– having only past exchange rates in the information set. Finally portfolio managers, the actors who actually buy and sell foreign assets, form their expectations as a weighted average of the predictions of fundamentalists and chartists. Portfolio managers update the weights over time in a rational, Bayesian manner, according to whether the fundamentalists or the chartists have recently been doing a better job of forecasting. Thus each of the three is acting rationally subject to certain constraints. The model departs from the orthodoxy in that the agents could do better, in expected value terms, if they knew the complete model. The departure is a general model of exchange rate determination

$$s_t = c\Delta s^m_{t+1} + z_t, \qquad c \geq 0, \tag{6}$$

where s_t is the log of the spot exchange rate, Δs^m_{t+1} is the rate of depreciation expected by the market, i.e. by the portfolio managers, and z_t, represents market fundamentals. Portfolio managers use a *weighted average of the expectations of fundamentalists and chartist*:

$$\Delta s^m_{t+1} = \omega_t \Delta s^f_{t+1} + (1-\omega_t)\Delta s^c_{t+1}, \qquad 0 \leq \omega_t \leq 1. \tag{7}$$

Fundamentalists' forecast are given by

$$\Delta s^f_{t+1} = v(\bar{s} - s_t), \tag{8}$$

where $\bar{s}$ is the fundamental exchange rate and v is the speed of adjustment. For simplicity, Frankel and Froot (1990b) assume that the 'chartists' believe that the exchange rate follows a random walk, that is,

$$\Delta s^c_{t+1} = 0. \tag{9}$$

Portfolio managers' expected change of exchange rates (7) then simplifies to

$$\Delta s^m_{t+1} = \omega_t v(\bar{s} - s_t). \tag{10}$$

The weight ω_t attached to fundamentalists views by portfolio managers evolves according to

$$\Delta\omega_t = \delta(\widehat{\omega}_{t-1} - \omega_{t-1}), \qquad 0 \leq \delta \leq 1, \tag{11}$$

where $\widehat{\omega}_{t-1}$ is defined as the weight, computed ex post, that would have perfectly predicted the realized change in the spot rate, that is, $\widehat{\omega}_{t-1}$ is defined by the equation

$$\Delta s_t = \widehat{\omega}_{t-1} v(\bar{s} - s_{t-1}). \tag{12}$$

Equations (11) and (12) together determine the change of weights that portfolio managers give to fundamentalist's views:

$$\Delta\omega_t = \delta\left(\frac{\Delta s_t}{v(\bar{s} - s_{t-1})} - \omega_{t-1}\right), \tag{13}$$

where the coefficient δ measures the speed of adaption. Portfolio managers thus adapt the weight given to the fundamentalist forecast in the direction of the weight that would have yielded a perfect forecast.

Frankel and Froot (1990a) take a continuous time limit and obtain differential equations for $\omega(t)$ and $s(t)$. Since the fundamental steady state may be unstable, the model is extended by adding an endogenous stabilizing fundamental force, due to current account imbalance when the exchange rate moves too far away from the fundamental. Simulations of the extended model show that the exchange rate may exhibit a temporary bubble, during which fundamentalists weight is driven to zero, with a rapidly increasing exchange rate, but at some point when the exchange rate has moved too far away from its fundamental value external deficits turn the trend and portfolio managers start giving more weight again to fundamentalists forecast, accelerating the depreciation. Frankel and Froot (1990a, p. 113) note that *"Ironically, fundamentalists are initially driven out of the market as the dollar appreciates,* even though they are ultimately right about its return to $\bar{s}$".

In the model the three types of agents, portfolio managers, chartists and fundamentalists are not fully rational. In defending their approach against the Friedman hypothesis that speculative, destabilizing investors will be driven out of the market by smart, stabilizing investors, Frankel and Froot (1986, pp.35-36) use a bounded rationality defense for their model [emphasis added]:

"All this comes at what might seem a high cost: portfolio managers behave irrationally in that they do not use the entire model in formulating their exchange rate forecasts. But another interpretation of this behavior is possible in that portfolio managers are actually doing the best they can in a confusing world. *Within this framework they cannot have been more rational; abandoning fundamentalism more quickly would not solve the problem in the sense that their expectations would not be validated by the resulting spot process in the long run. In trying to learn about the world after a regime change, our portfolio managers use convex combinations of models which are already available to them and which have worked in the past. In this context, rationality is the rather strong presumption that one of the prior models is correct. It is hard to imagine how agents, after a regime change, would know the correct model."*

3 Noise traders and behavioral finance

The work on HAMs discussed in this chapter is closely related to recent ideas from *behavioral finance*. In their recent survey, Barberis and Thaler (2003, p1052) state: *behavioral finance argues that some financial phenomena can plausibly be understood using models in which some agents are not fully rational*. Behavioral finance has two building blocks. The first is *limits to arbitrage*, meaning that it can be difficult and risky for rational arbitrageurs to correct mispricing caused by non-rational traders, because the mispricing may get worse in the short run when a majority of traders adopts a trend following strategy. The second building block is *market psychology*, an attempt to characterize which heuristics and biases play a role in financial markets. The financial market HAMs discussed in the current chapter fit within behavioral finance in that they provide tractable, parsimoniously parameterized models capturing key features in behavioral finance.

In the HAMs with fundamentalists versus chartists discussed in Section 2, none of the two trader types is fully rational, because none of the two takes into account the presence of the other. Would not, as the Friedman hypothesis suggests, a fully rational trader perform better and drive out all other trader types? In this section we discuss two early models due to DeLong, Shleifer, Summers and Waldmann (1990a,b). This approach has been called the *noise trader approach* and has e.g. been nicely summarized in Shleifer and Summers (1990). Another early, related HAM with “smart money” versus “ordinary” traders has been introduced by Shiller (1984). In these models there are two types of investors: *rational arbitrageurs* and *noise traders*. Arbitrageurs –also called *smart money* traders or *rational speculators*– are investors who form fully rational expectations about security returns. In contrast, “noise traders”, a term due to Kyle (1985) and Black (1986), –sometimes also called *liquidity traders*– are investors whose changes in asset demand are not caused by news about economic fundamentals but rather by non-fundamental considerations such as changes in expectations or market sentiment.

3.1 Rational versus noise traders

In DeLong et al. (1990a) there are two types of traders, noise traders and sophisticated, rational traders. There are two assets, a safe asset paying a fixed dividend r in each period, and a risky asset paying an uncertain dividend

$$r + \epsilon_t, \tag{14}$$

where ϵ_t is IID, normally distributed with mean 0 and variance σ_ϵ^2. The price of the unsafe asset in period t is denoted by p_t.

Noise traders incorrectly believe that they have special information about the future price of the risky asset. For example, they use signals from technical analysts, stock brokers or economic consultants and irrationally believe that these signals carry information and select their portfolios based upon these incorrect beliefs. For sophisticated traders it is optimal to exploit noise traders misperceptions. Sophisticated traders buy (sell) when noise traders depress (push up) prices. This contrarian trading strategy pushes prices in the direction of the fundamental value, but not completely.

For both trader types, demand for the risky asset is derived from expected utility maximization of constant absolute risk aversion utility of tomorrow's wealth,

$$\lambda_t^R = \frac{r + E_t p_{t+1} - (1+r)p_t}{2\gamma(\sigma^2_{p_{t+1}} + \sigma^2_\epsilon)}, \tag{15}$$

$$\lambda_t^N = \frac{r + E_t p_{t+1} - (1+r)p_t}{2\gamma(\sigma^2_{p_{t+1}} + \sigma^2_\epsilon)} + \frac{\rho_t}{2\gamma(\sigma^2_{p_{t+1}} + \sigma^2_\epsilon)}, \tag{16}$$

where γ is the coefficient of absolute risk aversion, $E_t[p_{t+1}]$ is the expected price at date $t+1$ conditional on information up to time t, $\sigma^2_{p_{t+1}}$ is the expected one period variance of p_{t+1} and ρ_t is the *misperception* of the expected price for tomorrow by the noise trader. Notice that the only difference in the demand of rational and noise traders is the second term in (16), due to the misperception ρ_t of the noise traders of next periods price of the risky asset. The misperception of noise traders is an *exogenously given* IID normally distributed random variable with mean ρ^* and variance σ^2_ρ.

There is a fixed fraction μ of noise traders and a fraction $1-\mu$ of rational, sophisticated traders. Market equilibrium requires that aggregate demand equals fixed supply normalized to 1, yielding the equilibrium price

$$p_t = \frac{1}{1+r}[r + E_t p_{t+1} - 2\gamma({}_t\sigma^2_{p_{t+1}} + \sigma^2_\epsilon) + \mu\rho_t]. \tag{17}$$

De Long et al. (1990a) only consider steady state equilibria satisfying the pricing rule

$$p_t = 1 + \frac{\mu\rho^*}{r} + \frac{\mu(\rho_t - \rho^*)}{1+r} - \frac{(2\gamma)}{r}[\sigma^2_\epsilon + \frac{\mu^2\sigma^2_\rho}{(1+r)^2}]. \tag{18}$$

The last three terms show the impact of noise traders on the price of the risky asset. Notice that, when the distribution of the misperception ρ_t of the noise traders converges to a point mass at $\rho^* = 0$, the price of the risky asset converges to its fundamental value $1 - (2\gamma\sigma^2_\epsilon/r)$. The second term on the RHS of (18) captures the fluctuations in prices due to the average misperception ρ^* of noise traders. The higher the average misperception of noise traders, the higher the asset price in equilibrium. The third term on the RHS of (18) captures the fluctuations in prices due to the variation $\rho_t - \rho^*$ in misperception of noise traders. When noise traders in period t are more bullish (bearish) than on average, the asset price increases (decreases). The final term on the RHS of (18) captures both *fundamental risk* and *noise trader risk*. A higher variance σ^2_ϵ, or a higher fraction μ of noise traders or a higher variance σ^2_ρ of noise traders' misperceptions all increase the risk premium to hold the risky asset and thus lower the asset price.

An important question is which type of traders, sophisticated or noise traders, earn relative higher returns. DeLong et al. (1990a) compute the (unconditional) *expected difference of return* between noise traders and sophisticated traders to be

$$E[\Delta R_t] = \rho^* - \frac{(\rho^*)^2 + \sigma^2_\rho}{2\gamma[\frac{\mu\sigma^2_\rho}{(1+r)^2} + \frac{\sigma^2_\epsilon}{\mu}]}. \tag{19}$$

From this expression it follows that for the noise traders to earn higher expected returns. the mean misperception ρ^* of returns must be positive. It is also clear, due to the dominating quadratic term in ρ^*, that for high values of ρ^* the expected difference in returns will become negative. However, for intermediate degrees of average bullishness ρ^* noise traders earn higher expected returns than sophisticated traders. Furthermore, the larger is the value of γ, that is, the more risk averse traders are, the larger is the range of ρ^*-values for which noise traders earn higher expected returns.

Imitation of beliefs

The arguments above show that when the fractions of both types are fixed, noise traders may earn higher expected returns suggesting that they may be able to survive in the long run. DeLong et al. (1990a) also discuss a dynamic version of the model with time varying fractions. Strategy selection is based upon the *relative performance* of the two strategies. Letting μ_t be the fraction of noise traders and R_t^N and R_t^S be the realized return of noise traders and sophisticated traders, the fraction of noise traders changes according to

$$\mu_{t+1} = \max\{0, \min[1, \mu_t + \alpha(R_t^N - R_t^S)]\}, \tag{20}$$

where $\alpha > 0$ is the rate at which investors become noise traders. According to (20) the strategy that has performed better, according to realized returns, attracts more followers, and such a rule may be interpreted as an *imitation* rule. It should be noted that this HAM with sophisticated agents and time varying fractions can only be solved for small values of α, because sophisticated agents have to calculate the effect of the realization of returns on the fractions of noise traders and sophisticated traders in the next period. For α sufficiently small realized returns can be calculated under the approximation that the fraction of noise traders remains the same.

For α small, the expected return difference between noise traders and sophisticated traders is obtained from (19) by replacing μ by μ_t:

$$E[\Delta R_t] = \rho^* - \frac{(\rho^*)^2 + \sigma_\rho^2}{2\gamma[\frac{\mu_t \sigma_\rho^2}{(1+r)^2} + \frac{\sigma_\epsilon^2}{\mu_t}]}. \tag{21}$$

The fraction of noise traders will increase (decrease) as long as the difference in expected returns (21) is positive (negative). A steady state fraction μ^* must satisfy either

$$E[\Delta R_t] = 0, \tag{22}$$

or $\mu^* = 0$ or $\mu^* = 1$. A straightforward computation shows that the number of steady states μ^* depends upon the parameter condition

$$\sigma_\epsilon^2 > \frac{(1+r)^2(\rho^* + \sigma_\rho^2)^2}{16\gamma^2(\rho^*)^2\sigma_\rho^2}. \tag{23}$$

The dynamics of the fraction of noise traders, in the limit as the speed of adjustment α tends to 0, has the following properties:

- If (23) is satisfied, then there are no steady states μ^* satisfying (22); noise traders always earn higher expected return and drive out sophisticated rational traders, that is, the noise trader share μ_t tends to 1;

- If (23) is not satisfied, then (22) has (at least one) positive real root(s); the smallest $\mu_L^* > 0$ is stable and thus a positive share of noise traders always survives in the market; if $\mu_L \geq 1$, then noise traders drive out sophisticated rational traders.

The fact that noise traders may survive in the long run, is only true if selection of trading strategies is based upon *realized returns*, and it can be argued that this contradicts traders' objective of maximizing expected utility. Since sophisticated investors maximize true expected utility, any other trading strategy that earns a higher mean return must have a variance sufficiently higher to make it non-optimal, that is, it must have sufficiently higher risk. When strategy selection is based upon *realized utility* instead of realized return, noise traders can not survive in the long run, because on average realized utility of sophisticated traders is higher than realized utility of noise traders. De Long et al. (1990a, p.724) however argue that a wealth based performance rule such as realized returns may be more relevant for real markets: *"... we find it plausible that many investors attribute the higher returns of an investment strategy to the market timing skills of its practitioners and not to its greater risk. This consideration may be particular important when we ask whether individuals change their own investment strategies that have just earned them a high return. When people imitate investment strategies, they appear to focus on standard metrics such as returns relative to market averages and do not correct for ex ante risk. As long as enough investors use the pseudo signal of realized returns to choose their own investment strategy, noise traders will persist"*. Realized returns are also important simply because those who make them become wealthier and get more weight in the market. The noise trader model thus contradicts the Friedman hypothesis.

3.2 Informed arbitrage versus positive feedback trading

DeLong, Shleifer, Summers and Waldmann (1990b) consider a different model where noise traders are replaced by *positive feedback traders*. The purpose of the model is to show that, in contrast to the Friedman hypothesis, in the presence of positive feedback traders, rational speculation can be *destabilizing*. The model only has four periods (0, 1, 2 and 3) and two assets, cash and stock. The stock is liquidated and pays an uncertain dividend $\Phi + \theta$ in period 3, when investors consume all their wealth. θ is normally distributed with mean 0, and no information about θ is revealed. Φ can take three different values, $-\phi$, 0 or ϕ; the value of Φ becomes public in period 2, and a signal ϵ about Φ is released in period 1.

There are three types of investors. *Positive feedback traders*, whose asset demand depends upon the latest observed price change, *informed rational speculators* who maximize utility of period 3 consumption using private information and *passive investors* whose asset demand depends only on the asset price relative to its fundamental price and who only have access to public information. In period 2, the value of Φ is revealed to both the informed rational investors and the passive investors. In period 1 a signal about period 2 fundamental news Φ is given, but only the informed rational investors have access to this private information. The fractions of the three types are constant over time. The fraction of positive feedback traders is normalized to 1, the fraction of rational informed speculators is μ and the fraction of passive investors is $1 - \mu$. The sum of the last two types is held constant in order to keep the risk-bearing capacity of the market constant. An increase in μ is therefore an increase in the proportion of rational investors who receive information and exploit short run price dynamics, holding the risk-bearing capacity of the market constant.

The structure of the model is summarized in Table 2. Informed rational speculators are perfectly rational in the sense that they form their demand optimally from mean-variance maximization given private information and taking into account the other type of investors in the market. Demand of passive investors is assumed to be negatively related to the price deviation from the fundamental. Finally, the demand of feedback traders is determined by the most recently observed past price change[3].

Structure of the Model				
		Total Demand of		
Period	Event	Positive Feedback Traders	Passive Investors	Informed Rational Speculators
0	None, benchmark period	0	0	optimal choice (=0)
1	speculators receive signal ϵ of period 2 fundamental shock	0	-αp_1	optimal choice
2	passive investors learn Φ	$\beta(p_1 - p_0)$	$-\alpha(p_2 - \Phi)$	optimal choice
3	Liquidation of stock dividend $\Phi + \theta$ revealed publically	$\beta(p_2 - p_1)$	$-\alpha(p_3 - (\Phi + \theta))$	optimal choice

Table 2: as in DeLong et al. (1990b, p385). Demand of different investor types and information for different investor types. β and α are the slope of the demand curves of positive feedback traders and passive investors. p_0, p_1, p_2 and p_3 are asset equilibrium prices in the corresponding periods.

In period 1, the rational informed investors receive a signal $\epsilon \in \{-\phi, 0, \phi\}$. We focus on the situation where this signal is positive, i.e. $\epsilon = \phi > 0$. We consider two cases, one where the signal is *noiseless* and a second case where the signal, $\epsilon = \phi$ is *imperfectly informative* and $\Phi = \phi$ or $\Phi = 0$ each with probability $1/2$. The equilibrium prices in periods 0, 1, 2 and 3 can be computed by solving the model backwards, and they are graphically represented in Figure 3. Period 0 forms a reference period and the initial price is set to the fundamental price 0, i.e. $p_0 = 0$. When there are no rational informed speculators (i.e. $\mu = 0$), the equilibrium price jumps from 0 in period 1 to ϕ in period 2, when private information becomes public. When there are rational informed traders in the market, arbitrage pushes up the equilibrium prices p_1 and p_2 in periods 1 and 2, in both the noiseless and noisy signal cases. This effect is amplified by the presence of positive feedback trading leading to equilibrium prices far above fundamental prices reflecting private information in period 1 and public information in period 2. The conclusion is that, in contrast to the Friedman hypothesis, *in the presence of positive feedback traders, rational speculation can be destabilizing*. The model thus explains overreaction to news about economic fundamentals, caused by rational informed speculators taking into account the presence of feedback traders.

[3] As pointed out in DeLong et al. (1990b, p385, footnote 6) it is the responsiveness of feedback traders to *past* price changes and not the responsiveness to current price changes that leads to the possibility of destabilizing rational speculation.

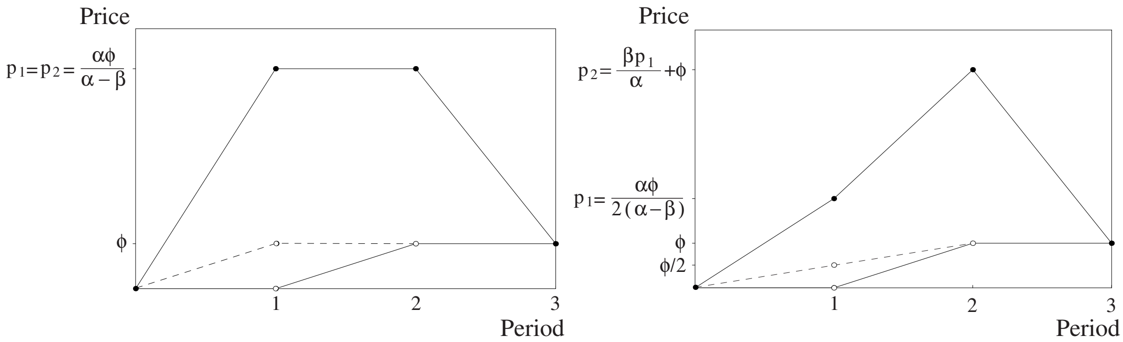

Figure 3: as in DeLong et al. (1990b, p390). Equilibrium prices with a noiseless signal (left panel) and a noisy signal (right panel) ϵ for rational informed traders. Without rational informed traders in the market (circles, lower graphs) the equilibrium price jumps from 0 in period 1 to ϕ in period 2 when the positive, private information becomes public. In the absence of feedback traders, arbitrage of rational informed traders pushes up the equilibrium prices p_1 to a fundamental value ϕ when the signal is perfect (left panel, dotted line) resp. a fundamental value $\phi/2$ (when all agents are informed, i.e. $\mu = 1$) when the signal is imperfect (right panel, dotted line). In the presence of feedback traders rational speculation by informed traders causes the equilibrium prices (squares, upper graphs) to overshoot the fundamental price by a large amount. α is the slope of the demand curve of the passive and informed rational traders; β is the responsiveness of positive feedback traders to past price changes.

In the models of DeLong et al. (1990a,b) the behavior of noise traders is exogenously given, and the other group, the sophisticated (informed) traders, take the presence of noise traders into account and respond perfectly rational to their erroneous behavior. In a way, this requires even more rationality than in a RE-model, because in a heterogeneous market a rational agent must anticipate the beliefs of all other, *non-rational* traders. More recently, behavioral finance HAMs have been developed where two (or more) different groups of *boundedly rational* traders interact. A recent example is Hong and Stein (1999), who consider a model with *newswatchers* versus *momentum* traders. Newswatchers make forecasts based on private information without conditioning on past prices, whereas momentum traders' forecasts are based on the most recent price change. These type of behavioral finance models can explain important stylized facts, which can not be explained by a perfectly rational agent EMH model, such as excess volatility, positive correlations of returns at short horizons and negative correlation of returns at long horizons. It also provides an explanation for the risk premium puzzle: because of noise trader risk, the difference between average returns on stocks and bonds –the risk premium– is higher than the fundamental risk. We refer the reader to the recent survey by Barberis and Thaler (2003) and their many references.

4 Complex dynamics

In the seventies and the eighties, due to the discovery of *deterministic chaos*, it became widely known that simple nonlinear deterministic laws of motion can generate seemingly unpredictable, chaotic fluctuations; see e.g. Medio and Lines (2001) for a mathematical introduction. Dynamic HAMs are often highly nonlinear, for example due to the fact that the weights or the fractions of the different trader types are time dependent. HAMs can therefore generate complicated, chaotic fluctuations for a broad range of parameter settings. Chaotic models offer the possibility to describe erratic, unpredictable movements in asset prices by a simple, nonlinear 'law of motion', and this possibility has stimulated much research in this area. In particular, a chaotic HAM with chaotic asset price fluctuations around a benchmark fundamental may explain excess volatility. In a non-linear, chaotic market system arbitrage trading is difficult and risky, because such a system is difficult to predict, especially when it is buffeted with (small) noise representing e.g. news about economic fundamentals. In this section, we review some nonlinear HAMs exhibiting periodic and chaotic asset price fluctuations.

In the models in subsections 4.1 and 4.2 the price setting mechanism is not the classical Walrasian market clearing framework, but rather a market maker who sets prices according to aggregate excess demand. Subsection 4.1 discusses a continuous time model due to Beja and Goldman (1980) and Chiarella (1992), allowing for limit cycles, whereas subsection 4.2 discusses a discrete time model due to Day and Huang (1990), exhibiting chaotic asset price fluctuations, and market maker model due to Farmer (2002) and Farmer and Joshi (2002). Finally, subsection 4.3 discusses an exchange rate model with fundamentalists and chartists of DeGrauwe, Dewachter and Embrechts (1993), with the weights of both trader types changing endogenously over time.

4.1 An early disequilibrium model with speculators

Beja and Goldman (1980) were among the first to consider a dynamic HAM with a stylized representation of the market institution by a *market maker* who adjusts prices according to aggregate excess demand. They argue that a real asset market does not operate as a perfect Walrasian market, but that a price formation process admitting a finite adjustment speed that allows for transactions at disequilibrium prices is a more accurate description. In their model traders try to exploit these market imperfections and, at least partly, act on their perception of the current price trend.

Movements in the asset price p are driven by aggregate excess demand with a finite adjustment speed, i.e. the price change is given by

$$\frac{dp}{dt} = D_t^f + D_t^c, \tag{24}$$

where D_t^f and D_t^c represent excess demand of fundamentalists and chartists respectively. Let $w(t)$ denote the (exogenously generated) fundamental price that clears fundamental demand at time t. Fundamental excess demand is assumed to be a linear function of the form

$$D_t^f = a(w(t) - p(t)), \qquad a > 0, \tag{25}$$

where the coefficient a measures the relative impact of fundamental demand upon price movements.

Let $\psi(t)$ be the chartists' assessment of the current price trend, and $g(t)$ the (exogenously given) return on alternative securities (e.g. $\psi(t)$ could represent the return on stocks and $g(t)$ the return on bonds). Chartists' excess demand is a linear function of the expected return differential $\psi(t) - g(t)$, that is,

$$D_t^c = b(\psi(t) - g(t)), \qquad b > 0, \tag{26}$$

where the coefficient b measures the relative impact of speculator's demand upon price movements. According to (24-26), aggregate price change is given by

$$\frac{dp}{dt} = a[w(t) - p(t)] + b[\psi(t) - g(t)] + e(t), \tag{27}$$

where $e(t)$ denotes an additional noise term. Speculators use an adaptive process for trend estimation

$$\frac{d\psi}{dt} = c[\frac{dp}{dt} - \psi(t)], \qquad c > 0, \tag{28}$$

where c is the adaption speed. The trend estimate ψ is thus adjusted upwards (downwards) when the current price change is higher (lower) than expected.

A stability analysis of the 2-D linear system of differential equations (27-28) shows that the system is stable if and only if $a > c(b-1)$. Hence, if the impact a of fundamental demand is sufficiently large or if the impact b of speculative demand is low ($b < 1$), then the market will be stable. However, when the market impact b of speculative demand becomes large and/or when the adaption speed c with which speculators adapt their perceived price trend becomes large, the system becomes unstable with exploding price oscillations. This simple, behavioral model thus shows that, under a market maker scenario, speculative trading may destabilize prices.

Chiarella (1992) considers a nonlinear generalization of the model, where linear chartists' excess demand (26) is replaced by a *nonlinear* function $h(\cdot)$ of the expected return differential $\psi(t) - g(t)$, that is,

$$D_t^c = h(\psi(t) - g(t)). \tag{29}$$

The function h is nonlinear, increasing and S-shaped. More precisely, h satisfies (i) $h'(x) > 0$, (ii) $h(0) = 0$, (iii) there exists x^* such that $h''(x) < 0$ (> 0) for all $x > x^*$ $(x < x^*)$, and (iv) $\lim_{x \to \pm\infty} h'(x) = 0$. Although Chiarella (1992) does not provide a micro-foundation for this aggregate excess demand function of chartists, he does provide arguments why such a demand function may be reasonable. For example, each chartist may seek to allocate a fixed amount of wealth between speculative risky assets and riskless bonds so as to maximize intertemporal utility of consumption. The demand for the risky asset is then proportional to the difference in expected return $\psi - g$, but is also bounded above and bounded below due to wealth constraints. For a chartist, the individual demand function would then be piecewise linear, and adding many such individual demand functions together leads approximately to an S-shaped increasing aggregate excess demand function.

Chiarella (1992) focuses on the simplest case where the fundamental price and the return on alternative investments are constant, $w(t) \equiv w$ and $g(t) \equiv g$. The dynamics of the nonlinear model is then described by the 2-D system of differential equations

$$\frac{dp}{dt} = a[w - p(t)] + h(\psi(t) - g) \tag{30}$$

$$\frac{d\psi}{dt} = c[\frac{dp}{dt} - \psi(t)]. \tag{31}$$

The nonlinear system has a unique steady state $(p^*, \psi^*) = (w + h(-g)/a, 0)$. The local stability analysis yields the same results as in Beja and Goldman (1980): a large market impact of speculative demand (i.e. a large $h'(-g)$) and/or a high adaption speed c with which speculators adapt their perceived price trend *destabilizes* the system. Moreover, Chiarella (1992) shows that in the unstable case, a (unique) *stable limit cycle* exists along which price and trend estimation of chartist fluctuate over time. The limit cycle and the corresponding time series are illustrated in Figure 4.

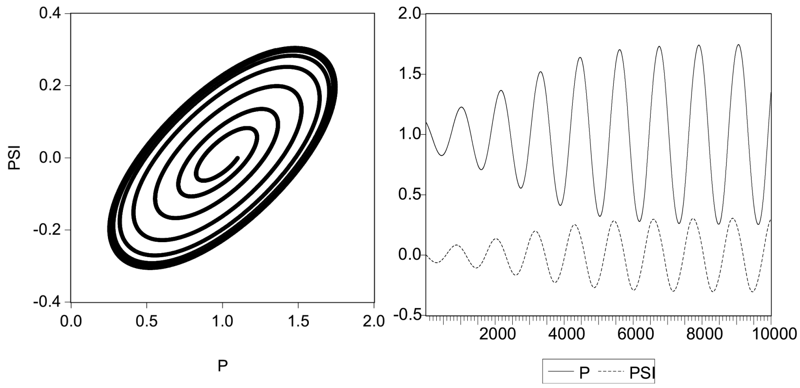

Figure 4: Limit cycle and time series of price (p) and perceived price trend (ψ) by chartists. When the perceived price trend is positive, the price change is reinforced by the speculators. The function $h(x) = \text{Tanh}(\lambda x)$, with $\lambda = 2$, and other parameters are $a = 0.5$, $w \equiv 1$, $g = 0$ and $c = 0.5$.

4.2 Market maker models

Another early, stimulating and influential model with price setting by a market maker has been introduced by Day and Huang (1990). The model is in discrete time and it is one of the first models exhibiting complicated, chaotic asset price fluctuations around a benchmark fundamental price, qualitatively similar to real stock market fluctuations, with bull markets suddenly interrupted by market crashes.

There are three types of investors, $\alpha-$investors, $\beta-$investors and market makers. The $\alpha-$investors base their investment decision upon a sophisticated estimate of the long run investment value u in relation to the current price and on an estimate of the chance for capital gains and losses. The $\alpha-$investors thus base their investment decision on a combination of (long run) economic fundamentals, such as dividends, earnings, growth, etc., and an estimate about the probability that an investment opportunity may disappear in the near future. The excess demand, D_t^α, by $\alpha-$investors as a function of the market price p_t is given by

$$\begin{array}{ll} D_t^\alpha = a(u - p_t)f(p_t), & \text{if } p \in [m, M], \\ \alpha(p) = 0 & \text{if } p < m \text{ or } p > M, \end{array} \tag{32}$$

where u is the (constant) long-run investment value expected by the $\alpha-$investors, a measures the relative strength of their investment demand, and $f(p)$ is a bimodal probability density with peaks near the extreme values m and M. The $\alpha-$investors believe that, when p_t is close to the topping price M, the probability of loosing a capital gain and experiencing a capital loss is high, and if p_t is close to the bottoming price m, the probability of missing a capital gain by failing to buy is high.

The $\beta-$investors are less sophisticated than the $\alpha-$investors. Their investment decision is based upon a simple extrapolative rule of their expected investment value, $u^s_{t+1} = p_t + \sigma(p_t - v)$, where v is the (constant) fundamental value of the asset. The $\beta-$investors thus believe that the investment value of the asset can be extrapolated from past deviations from the fundamental value. Excess demand of $\beta-$investors is given by

$$D^\beta_t = \delta(u^s_{t+1} - p_t) = b(p_t - v), \tag{33}$$

with $b = \delta\sigma$. Hence, $\beta-$investors buy (sell) when the price is above (below) its perceived fundamental value. In contrast to the $\alpha-$investors, $\beta-$investors do not take into account an estimate of the probability of investment opportunities in the near future.

The third trader type are *market makers* who mediate transactions on the market out of equilibrium by providing liquidity. The market maker sets a price and supplies stock out of his inventory when there is excess demand and accumulates stock to his inventory when there is excess supply. Aggregate excess demand of $\alpha-$ and $\beta-$investors is given by

$$ED(p_t) = D^\alpha_t + D^\beta_t, \tag{34}$$

and the change of the market makers' inventory V_t of stock is

$$V_{t+1} - V_t = -ED(p_t). \tag{35}$$

Prices are set by the market maker according to the price adjustment rule

$$p_{t+1} = g(p_t) := p_t + \lambda ED(p_t), \qquad \lambda > 0, \tag{36}$$

where the parameter λ is the *speed of adjustment*. This price adjustment rule is similar to the classical price tâtonnement process. Day and Huang (1990) argue that the price adjustment rule is determined by the market institution, and that the market maker should be viewed as a stylized version of the specialist at the New York Stock Exchange.

When the probability distribution $f(p)$ is bimodal, the price adjustment function g in (36) is a non-monotonic 1-D mapping. Day and Huang (1990) consider a simple example $f(p) = (p - m + \epsilon)^{-d_1}(M + \epsilon - p)^{-d_2}$, for $m \leq p \leq M$ and $f(p) = 0$ otherwise, whose graph is illustrated in Figure 5. They show that for suitable values of the parameters, stock prices exhibit chaotic fluctuations.

In these simulations, the fundamental value v and the long run investment value u are both constant and equal to 0.5. Stock prices switch irregularly between bull markets with prices rising above the fundamental and bear markets with prices dropping below the fundamental value. Prices are driven up (or down) by trend extrapolating $\beta-$investors, until they get close to their topping (or bottoming) price where the excess demand of $\alpha-$investors sharply decreases (increases) causing the bull (bear) market to end. The $\beta-$investors (who may be

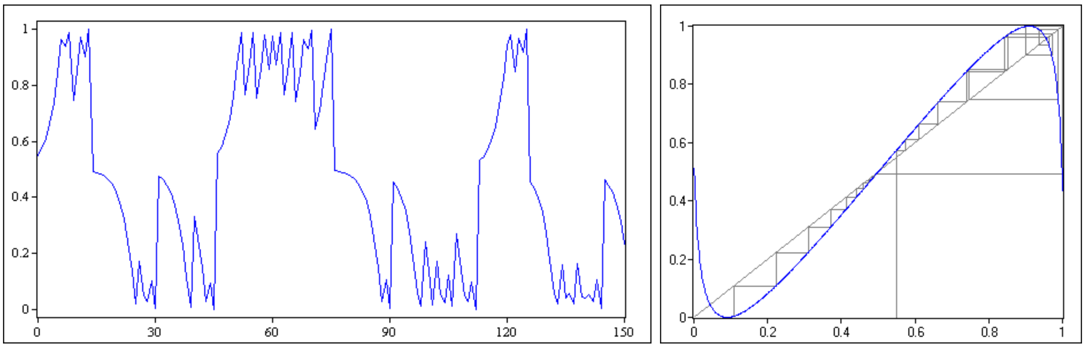

Figure 5: Chaotic time series, with an initial state p_0 just above the fundamental price 0.5, and graph of the 1-D map (36). Stock prices switch irregularly between bull and bear markets, as explained in the graphical analysis for initial state $p_0 = 0.55$ (right plot). Parameters: $m = 0$, $M = 1$, $d_1 = d_2 = 0.5$, $\epsilon = 0.0097$, $u = v = 0.5$, $a = 0.2$, $b = 0.88$ and $\lambda = 1$.

compared with noise traders) 'follow market prices like sheep' thus making the market for $\alpha-$investors (or better informed investors) whose behavior is exactly opposite. Day and Huang (1990, p.307) also note that "*the market makers must buy high from investors and sell low to them, but the damage to their position can be offset by investments on their own account and by their fees for conducting the market*".

More recently, Farmer (2002) and Farmer and Joshi (2002) have derived a similar price setting rule, which they call the *market impact function*. In their model, there are N *directional traders* who buy or sell a single risky asset by placing market orders, which are always filled. Typically, the buy and sell orders of the directional traders do not match, but the excess demand or excess supply is taken up from or added to the inventory of a *market maker*, who increases (decreases) the price when there is net excess demand (supply). The market impact function is the algorithm used by the market maker to set prices. To be more concrete, let $x^i_t = x^i(P_{t-1}, P_{t-2}, \cdots, I_{t-1})$ be the position of directional trader i at time t, where x^i represents the trading strategy or decision rule of agent i depending on past prices $P_{t-1}, P_{t-2}, \cdots$ and exogenous information I_{t-1}. The net order ω^i_t of directional trader i is given by

$$\omega^i_t = x^i_t - x^i_{t-1}. \tag{37}$$

The aggregate net order is then

$$\omega = \sum_{i=1}^{N} \omega^i. \tag{38}$$

The market maker adjust prices according to

$$P_{t+1} = P_t \phi(\omega), \tag{39}$$

with ϕ an increasing function and $\phi(0) = 1$. Taking logs a linear approximation yields

$$\log P_{t+1} - \log P_t \approx \frac{\omega}{\mu}, \tag{40}$$

where the parameter μ normalizes the order size and is called *liquidity*. The function ϕ is referred to as the *log linear market impact function*. Writing $p_t = \log P_t$ and adding a noise term ϵ_t (e.g. representing noise traders), the (log) price dynamics is given by

$$p_{t+1} = p_t + \frac{1}{\mu} \sum_{i=1}^{N} \omega^i(p_t, p_{t-1}, \cdots, I_t) + \epsilon_t. \tag{41}$$

Notice that this price updating rule is essentially the same as the market maker price adjustment rule in Beja and Goldman (1980), Day and Huang (1990) and Chiarella (1992), as discussed above, except that p_t now represents log price instead of price. The liquidity parameter μ in (41) is inversely related to the speed of adjustment λ in (36). Farmer (2002) and Farmer and Joshi (2002) consider different types of directional traders, either using value investment strategies (or fundamental trading strategies) based upon the perceived value of the asset or using chartists, trend following trading strategies based upon past prices. They show that trend following strategies induce short run positive autocorrelations in returns, whereas value trading induces negative autocorrelations. Furthermore, they present a simple HAM with value investors versus trendfollowers, where autocorrelations of returns are close to zero and other stylized facts observed in financial time series, such as noise amplification, excess volatility, excess kurtosis and clustered volatility, are also matched.

4.3 A chaotic exchange rate model

DeGrauwe, Dewachter and Embrechts (1993) introduce an equilibrium exchange rate model with fundamentalists and chartists, following earlier work of Frankel and Froot (1986,1990a). It is one of the first HAMs where the weights of the two investor types is determined *endogenously* and fluctuates over time. The basic equation determining the exchange rate is

$$s_t = X_t(E_t[s_{t+1}])^b, \tag{42}$$

where s_t is the exchange rate in period t, X_t is an exogenous variable representing the underlying economic fundamental driving the exchange rate, $E_t[s_{t+1}]$ is next period's expected exchange rate and the parameter b is a discount factor, $0 < b < 1$.

The aggregate change in the expected future exchange rate consists of two components, a forecast made by chartists and a forecast made by fundamentalists:

$$E_t[s_{t+1}]/s_{t-1} = (E_{ct}[s_{t+1}]/s_{t-1})^{m_t}(E_{ft}[s_{t+1}]/s_{t-1})^{1-m_t}, \qquad 0 \leq m_t \leq 1, \tag{43}$$

where $E_t[s_{t+1}]$ is the aggregate market forecast for next period's exchange rate made at date t, $E_{ct}[s_{t+1}]$ and $E_{ft}[s_{t+1}]$ are the forecasts made by chartists and fundamentalists, and m_t and $1 - m_t$ are the *weights* given to chartists and fundamentalists respectively.

Fundamentalists believe that the exchange rate returns towards its fundamental rate s^*_t at rate α, $0 \leq \alpha \leq 1$, that is,

$$E_{ft}[s_{t+1}]/s_{t-1} = (s^*_{t-1}/s_{t-1})^{\alpha}, \tag{44}$$

where $s^*_t = X_t^{1/(1-b)}$ is the *steady state* equilibrium exchange rate s^*_t obtained from (42). Chartists look for patterns in past exchange rates and their forecast is

$$E_{ct}[s_{t+1}] = f(s_{t-1}, s_{t-2}, ...s_{t-N}), \tag{45}$$

where N is the maximum lag used. DeGrauwe et al. (1993) mainly focus on moving average rules for chartists of the form

$$\frac{E_{ct}[s_{t+1}]}{s_{t-1}} = (\frac{SMA(s_{t-1})}{LMA(s_{t-1})})^{2\gamma}, \qquad \gamma > 0 \tag{46}$$

where $SMA(s_{t-1})$ and $LMA(s_{t-1})$ are short run and long run moving averages. According to (46), when the short run moving average is above (below) the long run moving average, chartists expect a future increase (decline) of the exchange rate. This type of technical trading rule is employed frequently by financial practitioners. The parameter γ measures the rate at which chartists extrapolate the past into the future. DeGrauwe et al. (1993) mainly focus on the simplest moving average rules, with a one-period change short run rule

$$SMA(s_{t-1}) = \frac{s_{t-1}}{s_{t-2}}, \tag{47}$$

and a simple two-period moving average for the long run, i.e.

$$LMA(s_{t-1}) = (\frac{s_{t-1}}{s_{t-2}})^{0.5}(\frac{s_{t-2}}{s_{t-3}})^{0.5}. \tag{48}$$

Using the short run and long run moving averages (47) and (48), the chartists expected change of the exchange rate (46) becomes

$$\frac{E_{ct}[s_{t+1}]}{s_{t-1}} = (\frac{s_{t-1}}{s_{t-2}})^{\gamma}(\frac{s_{t-3}}{s_{t-2}})^{\gamma}. \tag{49}$$

We now turn to the *endogenous* determination of the weight m_t of chartists. DeGrauwe et al. (1993) postulate the following weighting function:

$$m_t = \frac{1}{1 + \beta(s_{t-1} - s^*_{t-1})^2}, \qquad \beta > 0. \tag{50}$$

DeGrauwe et al. (1993, pp.75-76) present the following behavioral motivation. There is uncertainty about the fundamental exchange rate equilibrium and fundamentalists have heterogeneous expectations about its true value. When the exchange rate is at its fundamental equilibrium value, $s_{t-1} = s^*_{t-1}$, half of the fundamentalists will find that the market rate is too low, and the other half will find it too high compared to their own estimate. Assuming that all fundamentalists have the same degree of risk aversion and the same wealth, the amount of foreign exchange bought by the first half equals the amount sold by the second half. Hence, when the exchange rate equals its fundamental value, fundamentalists do not influence the market and the market expectation will be completely dominated by chartists ($m_t = 1$). When the exchange rate deviates from its fundamental equilibrium value, the weight of fundamentalists increases, at a rate measured by the parameter β. The endogenous switching mechanism (50) for the weights of chartists and fundamentalists acts as a "*far from the fundamental equilibrium stabilizing force*" on exchange rates. The more the exchange rate deviates from its fundamental equilibrium, the higher the weight of fundamentalists and the stronger the exchange rate will be pushed back towards its fundamental equilibrium value.

In the simplest case, with the fundamental $X_t \equiv 1$ normalized to 1, and one-period short run and two-period long run moving averages, the model can be written as

$$s_t = s^{\phi_1}_{t-1}s^{\phi_2}_{t-2}s^{\phi_3}_{t-3} \tag{51}$$

$$m_t = \frac{1}{1 + \beta(s_{t-1} - 1)^2}, \tag{52}$$

with $\phi_1 = b[1 + \gamma m_t - \alpha(1 - m_t)]$, $\phi_2 = -2b\gamma m_t$ and $\phi_3 = b\gamma m_t$. The unique fundamental steady state is $(s^*, m^*) = (1, 1)$. The model exhibits rich dynamical behavior ranging from a stable steady state to (quasi-)periodic as well as chaotic dynamics. In particular, when the parameter γ, measuring the rate at which chartists extrapolate a trend, is sufficiently large, the fundamental steady state becomes unstable and chaotic exchange rate fluctuations around the fundamental equilibrium rate arise, as illustrated in Figure 6. In the next Sections we discuss HAMs with switching between trading strategies driven by evolutionary selection and social interactions.

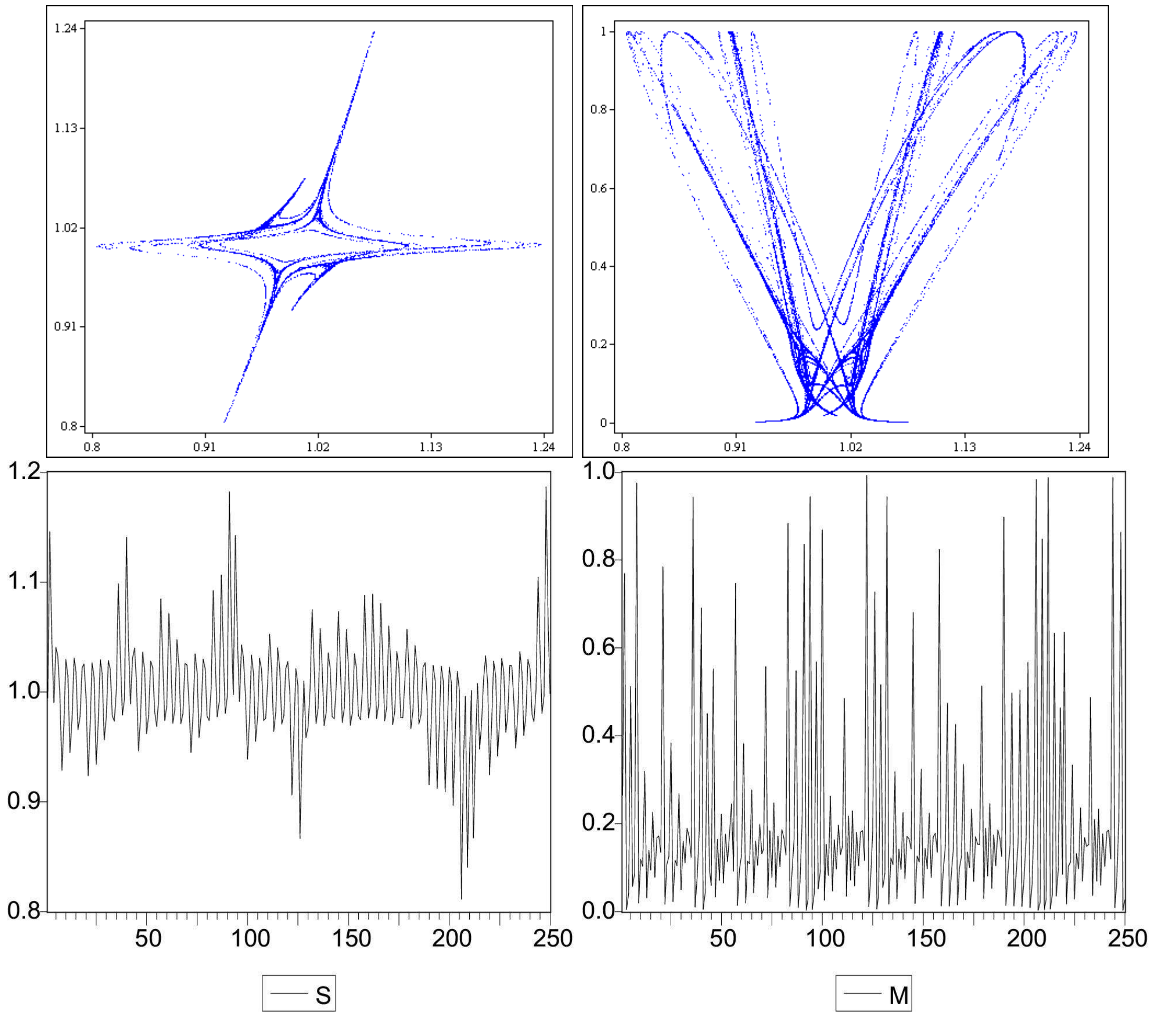

Figure 6: Strange attractor in the (s_t, s_{t-1}) (top left) and (s_t, m_t) (top right) phase space and corresponding chaotic time series of the exchange rate s_t (bottom left) and the weight of chartists m_t (bottom right). Parameters: $b = 0.95$, $\alpha = 0.65$, $\gamma = 3$ and $\beta = 10000$.

5 Interacting agents

In this section we discuss models in which individual agents interact stochastically. At first sight one may think that, due to a law of large numbers, stochastic interactions average out and can not affect aggregate variables. However, this is not the case. Even weak (local) interactions among individuals may lead to strong dependencies and cause large movements at the aggregate level. Aggregation of simple interactions at the micro level may generate sophisticated behavior and structure at the macro level. An early model with interaction effects has been introduced by Föllmer (1974), who considers an exchange economy with random preferences with a probability law depending upon the agents' environment. Using results on interacting particle systems in physics, Föllmer (1974) shows that even short range interaction may propagate through the economy and lead to aggregate uncertainty causing a breakdown of price equilibria. In this section we discuss work on local interactions by Kirman (1991,1993) and work on social interactions by Brock and Durlauf (2001ab)[4]. These papers have been quite influential and stimulated much work in this area. For surveys on interacting agent models see, for example, Brock (1993) and Kirman (1999); see also the papers on path dependence in Arthur (1994) and the collection of articles in Gallegati and Kirman (1999) and Delli Gatti et al. (2000).

5.1 An exchange rate model with local interactions

This section discusses an exchange rate model with fundamentalists and chartists introduced by Kirman (1991). The model consists of two parts: an equilibrium model of foreign exchange rate and a model of opinion formation as described by the stochastic model of recruitment proposed by Kirman (1993).

The stochastic recruitment model was motivated by an observed puzzle in biology concerning the behavior of ants. When ants face two different but identical food sources, surprisingly often the majority concentrates on one of the food sources, say with 80% of the population on one food source and only 20% of the populations on the other. Moreover, after some time these proportions suddenly switch. Ants facing a symmetric situation, thus collectively behave in an asymmetric way. Kirman (1993) proposed a simple and elegant dynamic stochastic model explaining this observed asymmetric, aggregate behavior. The model offers an explanation for the behavior of ants, but here we follow Kirman's discussion of the model within a financial market framework. There is a fixed number of N agents. Agents must form an opinion about next period's price p_{t+1} of a risky asset and can choose between two opinions, optimistic and pessimistic. The expectations of agents are affected by random meetings with other agents. The state of the system is determined by the number k of agents holding say the optimistic view, with $k \in \{0, 1, 2, ..., N\}$. Two agents meet at random. The first agent is converted to the second agent's view with probability $(1 - \delta)$. There is also a small probability ϵ that the first agent will change his opinion independently. This (small) $\epsilon-$probability is necessary, in order to prevent the system to get stuck in the absorbing extreme states $k = 0$

[4]Related continuous time diffusion models of stock prices with stochastic interacting agents have been pioneered by Föllmer and Schweizer (1993) and Föllmer (1994).

or $k = N$. The state k then evolves according to

$$
\begin{aligned}
&k \rightarrow k+1, \quad \text{with probability} \quad P(k,k+1) = (1-\frac{k}{N})(\epsilon + (1-\delta)\frac{k}{N-1}), \\
&k \rightarrow k-1, \quad \text{with probability} \quad P(k,k-1) = \frac{k}{N}(\epsilon + (1-\delta)\frac{N-k}{N-1}), \\
&k \rightarrow k, \qquad\quad \text{with probability} \quad P(k,k) = 1 - P(k,k+1) - P(k,k-1).
\end{aligned} \tag{53}
$$

The stochastic process (53) is a simple Markov chain. Kirman investigates the equilibrium distribution $\mu(k)$ of (53), and shows that the form of the equilibrium distribution $\mu(k)$ depends on the relative magnitude of the parameters δ and ϵ:

- if $\epsilon < (1-\delta)/(N-1)$, then the equilibrium distribution is bimodal, with a minimum at $k/N \approx 0.5$ and maxima at the extremes $k = 0$ and $k = N$;
- if $\epsilon = (1-\delta)/(N-1)$, then the equilibrium distribution is uniform;
- if $\epsilon > (1-\delta)/(N-1)$, then the equilibrium distribution is unimodal with a maximum at $k/N \approx 0.5$.

Note that this result does not depend on the size of the probabilities δ or ϵ itself, but rather on their relative magnitudes. When the probability ϵ of self-conversion is low compared to the probability $(1-\delta)$ of being converted by the other trader, the limiting distribution is bimodal with maxima at the extremes. In that case, a typical time series of the state k is highly persistent and spends little time close to its average $k = 0.5$, but much more time close to the extremes $k = 0$ and $k = N$, as illustrated in Figure 7. This equilibrium distribution thus explains the asymmetric 80%-20% distribution of ants and its occasional flipping to a 20%-80% distribution and vice versa.

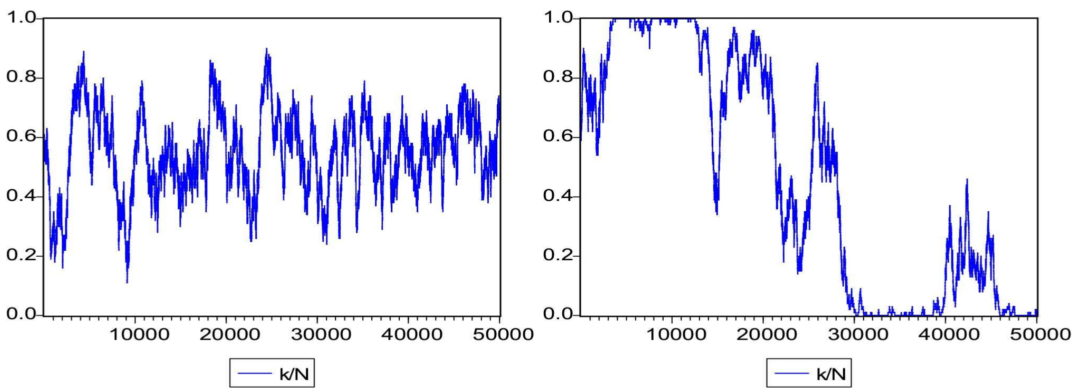

Figure 7: Fractions k/N of optimistic types, for $N = 100$, $\delta = 0.01$ and $\epsilon = 0.05$ (left plot) resp. $\epsilon = 0.002$ (right plot). In the latter case (right plot) the equilibrium distribution is bimodal with peaks at 0 and 1 and the time series is highly persistent and spends relatively much time close to the extremes. In the other case (left plot) the fraction stays relatively close to 0.5, i.e. to a symmetric distribution of the two types, most of the time.

Kirman (1991) considers an exchange rate model where the fractions of chartists and fundamentalists are driven by the stochastic model for opinion formation. The exchange rate equilibrium model is similar to the model of Frankel and Froot (1986), but Kirman (1991) provides a micro-foundation of asset demand. Agents can choose to invest in a risk free domestic currency paying a fixed interest rate r or in a risky foreign currency paying an uncertain (stochastic) dividend y_{t+1} in period $t+1$, assumed to be IID with mean $\bar{y}$. Agent type i maximizes expected utility from a mean-variance utility function $U^i(W^i_{t+1}) = E_i[W^i_{t+1}] - \mu^i V_i[W^i_{t+1}]$, where E_i and V_i denote agent type i's belief about conditional expectation and conditional variance of tomorrow's wealth W^i_{t+1} and $2\mu_i$ represents risk aversion. Agent type i's demand for foreign currency is then given by

$$d^i_t = \frac{s^e_{i,t+1} + \bar{y} - (1+r)s_t}{2\alpha\mu^i}, \tag{54}$$

where $s^e_{i,t+1}$ represents agent type i's expectation about the exchange rate s_{t+1}, $\bar{y}$ is the mean of the IID dividend process and $\alpha = V[s_{t+1} + y_{t+1}]$. In the case of homogeneous, rational expectations, the fundamental value of the exchange rate is given by

$$s^* = \frac{\bar{y} - 2\alpha\mu_i X_t}{r}, \tag{55}$$

where X_t is the supply of foreign exchange. In the heterogeneous agents case of fundamentalists versus chartists, with fractions n_t resp. $1 - n_t$, market equilibrium yields

$$n_t d^f_t + (1 - n_t)d^c_t = X_t. \tag{56}$$

The expectations of fundamentalists and chartists about next period's exchange rate s_{t+1} are given by[5]

$$s^e_{f,t+1} = s_{t-1} + v(s^* - s_{t-1}), \qquad 0 \le v \le 1, \qquad \text{fundamentalists} \tag{57}$$

$$s^e_{c,t+1} = s_{t-1} + g(s_{t-1} - s_{t-2})), \qquad g > 0, \qquad \text{chartists.} \tag{58}$$

Fundamentalists believe that the exchange rate will move back towards its fundamental value s^*, or equivalently, their expected change of the exchange rate is proportional to the observed distance to the fundamental. In the special case $v = 1$ fundamentalists expect the exchange rate to jump to its fundamental value s^* immediately, whereas the other extreme case $v = 0$ corresponds to naive expectations where fundamentalists expect the exchange rate to follow a random walk. Chartists extrapolate in a simple linear way and forecast the change of the exchange rate to be proportional to the latest observed change; Kirman focuses on the case $g = 1$.

Substituting the expectation rules (57) and (58) in the market equilibrium equation (56) and solving for the equilibrium exchange rate yields the difference equation

$$(1+r)s_t = [1 - vn_t + g(1-n_t)]s_{t-1} - g(1-n_t)s_{t-2} + \bar{y} - 2\alpha\mu^i X_t. \tag{59}$$

In deviations $x_t = s_t - s^*$ from the fundamental benchmark this simplifies to

$$(1+r)x_t = [1 - vn_t + g(1-n_t)]x_{t-1} - g(1-n_t)s_{t-2}. \tag{60}$$

[5] We choose a specification where s_{t-1} is the most recent observation used in the forecasts of s_{t+1}. Kirman discusses a specification where s_t is used as the most recent observation to forecast s_{t+1}, but in that case the HAM generates exploding exchange rate paths. As Kirman (1991, p.364) notes when expectations are based on earlier observations (such as s_{t-1}) the HAM allows for symmetric bubbles, both rising and falling. The approach chosen here is similar to the asset pricing model of Brock and Hommes (1998), as discussed in Section 8, and the forecasting rules (57) and (58) are the same as in Gaunersdorfer and Hommes (2005).

Kirman's exchange rate model with fundamentalists versus chartists is thus given by (60) with the fraction $n_t = k_t/N$ evolving according to the Markov chain (53)[6]. It is easily verified that when all agents are fundamentalists, i.e. $n_t \equiv 1$, (60) yields

$$x_t = \frac{1-v}{1+r} x_{t-1}, \tag{61}$$

which is a *stable* linear system with eigenvalue $\lambda = (1-v)/(1+r)$. In the other extreme case when all agents are chartists, i.e. $n_t \equiv 0$, (60) reduces to

$$x_t = \frac{1+g}{1+r} x_{t-1} - \frac{g}{1+r} x_{t-2}. \tag{62}$$

For $g = 1$ (62) has a pair of stable complex eigenvalues. Notice however that, if the time period of the model is one day, the daily domestic interest rate r is very close to 0 so that these complex eigenvalues are in fact close to a unit root +1. These complex roots become unstable when g increases beyond $1+r$, that is, when chartists expect the change in exchange rates to be larger than the risk free gross return.

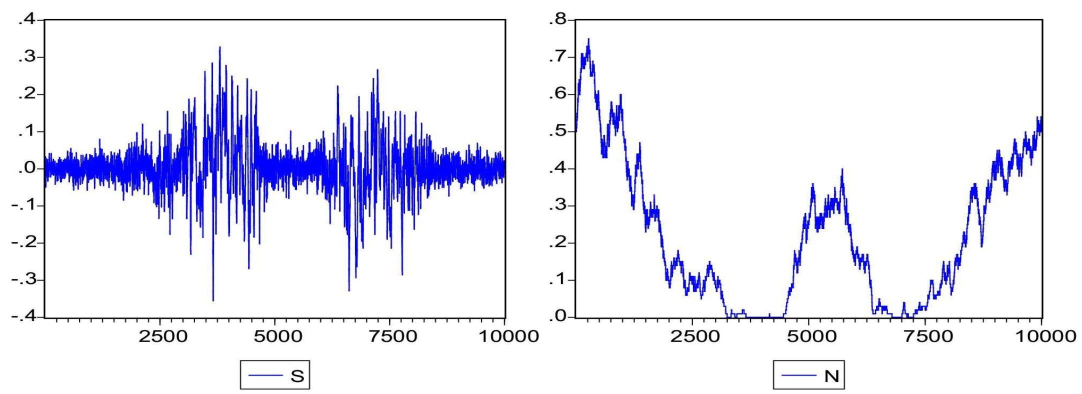

Figure 8: Time series of exchange rates (left plot) and fraction of fundamentalists (right plot). Parameters: $N = 100$, $\epsilon = 0.002$, $\delta = 0.01$, $r = 0.01$, $v = 0.5$ and $g = 0.8$. Exchange rates switch irregularly between phases of high volatility when the market is dominated by chartists and low volatility when the market is dominated by fundamentalists.

In periods when the market is dominated by fundamentalists, the exchange rate s_t is stable and is pushed towards its fundamental value s^*. In contrast, when the market is dominated by chartists, the exchange rate is driven by a stable, but near unit root process for $g = 1$ or an unstable process when $g > 1 + r$. A typical example of simulated time series of the exchange rate and the fraction of fundamentalists is illustrated in Figure 8. The fraction n_t of fundamentalists is driven by the stochastic recruitment model with the same parameters as in Figure 7 (right plot) and is therefore highly persistent, switching between two different phases where one of the two groups dominates the market. When chartists (fundamentalists) dominate the market, i.e. when n_t is close to 0 (1), volatility of the exchange rate fluctuations is high (low). This HAM therefore captures, at least qualitatively, the phenomenon of volatility clustering, with exchange rates switching irregularly between phases of high and

[6]Kirman (1991, pp.359-360) describes a slightly more complicated way of determining the fractions of the two types. Agents try to assess the majority opinion, but observe $n_t = k_t/N$ with noise. If agent i's observation $q_{it} = n_t + \epsilon_{it} \geq 1/2 (< 1/2)$, then he acst as a fundamentalist (chartist).

low volatility. Kirman and Teyssière (2002) discuss stylized facts, such as clustered volatility and long memory, generated by the model in more detail. Section 6 of this chapter also discusses stylized facts generated by HAMs. A related model with interaction through a random communication structure has been introduced by Cont and Bouchaud (2000); see also the survey of Kirman (1999).

5.2 Social interactions

Social interaction among individuals refers to a situation where the utility or payoff of an individual agent depends directly upon the choices of other individuals in their reference group, in addition to the dependence which occurs through the intermediation of markets. When the spillovers are positive, i.e. when the payoff is higher if others behave similarly, social interactions induce a tendency for conformity among members of the reference group. Social interactions may explain large cross-group variations, when different groups conform to alternative, self-reinforcing behavior. In the absence of a coordination mechanism social interactions can lead to multiple equilibria. Social interactions may cause a large social multiplier, meaning that small changes in private utility may cause large changes at the aggregate level.

Schelling (1971) introduced one of the first models with some form of social interaction. He considered a model where individuals have preferences over their neighborhood of racial composition and showed that, even when these preferences are relatively weak, it may lead to pronounced residual segregation. Brock and Durlauf (2001ab) have written excellent surveys on social interaction models in economics and developed a general class of social interaction models. A key feature of their models, following Brock (1993) and Blume (1993), is the use of discrete choice models with interaction effects. Their approach leads to analytically tractable models that can be used in estimating social interaction effects using the discrete choice framework of Manski and McFadden (1981). In this section we discuss a simple binary choice model with social interactions, closely following the presentation in Brock and Durlauf (2001a); the interested reader is referred to Brock and Durlauf (2001b) and their references for detailed discussions of more general social interactions models.

Each individual of a population of N agents makes a binary choice $\omega_i \in \{-1, +1\}$. Let $\omega_{-i} = (\omega_1, \dots \omega_{i-1}, \omega_{i+1}, \dots \omega_I)$ denote the choices of all agents other than i. Individual utility derived from choice ω_i consists of three components:

$$V(\omega_i) = u(\omega_i) + S(\omega_i, \mu_i^e(\omega_{-i})) + \epsilon(\omega_i). \tag{63}$$

Here $u(\omega_i)$ represents *private utility* associated with choice ω_i, $S(\omega_i, \mu_i^e(\omega_{-i}))$ represents *social utility* depending upon choice ω_i of individual i as well as upon the conditional probability measure $\mu_i^e(\omega_{-i})$ agent i places on the choice of other agents and $\epsilon(\omega_i)$ is an idiosyncratic random utility term IID distributed across agents. Instead of a general dependence of social utility on the conditional probability measure $\mu_i^e(\omega_{-i})$, it is often assumed that social utility depends upon agent i's expectation $\bar{m}_i^e$ of other individual choices j given by the average of subjective expected values $m_{i,j}^e$, i.e.

$$\bar{m}_i^e = \frac{1}{N-1} \sum_{j \neq i} m_{i,j}^e. \tag{64}$$

We focus on the case of *global interaction*, where the average (64) is taken over the entire population, i.e. over all individuals j different from i. One may also consider *local interaction* by restricting the average over the reference group of an individual i.

Brock and Durlauf (2001a) focus on simple and tractable parametric representations of both the social utility term and the probability density of the random utility term. Assuming constant cross partial derivatives leads to two functional forms for social utility. The first form is given as a multiplicative interaction between individual and expected average choices, that is,

$$S(\omega_i, \bar{m}_i^e) = J\omega_i\bar{m}_i^e. \tag{65}$$

This form is referred to as *proportional spillover*, because the percentage change in individual utility from a change in the mean choice level is constant. The second parameterization of social utility with constant partial derivatives captures the pure conformity effect as considered by Bernheim (1994), and is given by

$$S(\omega_i, \bar{m}_i^e) = -\frac{J}{2}(\omega_i - \bar{m}_i^e)^2. \tag{66}$$

Notice that this form penalizes choices far from the mean more strongly than the proportional spillover case. Using the fact that $\omega_i^2 = 1$, (66) can be rewritten as

$$-\frac{J}{2}(\omega_i - \bar{m}_i^e)^2 = J\omega_i\bar{m}_i^e - \frac{J}{2}(1 + (\bar{m}_i^e)^2). \tag{67}$$

This shows that (65) and (66) only differ in levels, but coincide on the terms including individual choices. Therefore, these two different parameterizations of social utility lead to the same discrete choice probabilities as discussed below. In what follows we focus on the proportional spillover specification for social utility in (65).

A standard way to obtain a convenient parameterization for the choice probabilities is to assume that the random utility terms $\epsilon(-1)$ and $\epsilon(1)$ in (63) are independent and extreme-value distributed, so that the difference in errors are logistically distributed,

$$\text{Prob}\{\epsilon(-1) - \epsilon(1) \leq x\} = \frac{1}{1 + \exp(-\beta x)}, \qquad \beta \geq 0. \tag{68}$$

Under this assumption the probability for individual choices is given by the logit model probability

$$\text{Prob}\{\omega_i\} = \frac{\exp(\beta[u(\omega_i) + J\omega_i\bar{m}_i^e])}{\sum_{\nu_i\in\{-1,1\}}\exp(\beta[u(\nu_i) + J\nu_i\bar{m}_i^e])}. \tag{69}$$

The parameter β is called the *intensity of choice* and it is inversely related to the level of random utility $\epsilon(\omega_i)$. In the extreme case $\beta = \infty$ the random utility term will vanish and all agents will choose the alternative with highest utility. In the other extreme case $\beta = 0$ the effect of the random utility term will dominate both individual and social utility and each alternative will be chosen with probability $1/2$.

Since the errors $\epsilon(\omega_i)$ are independent across agents, the joint probability distribution over all choices is given by

$$\text{Prob}\{\omega\} = \frac{\exp(\beta[\sum_{i=1}^N(u(\omega_i) + J\omega_i\bar{m}_i^e)])}{\sum_{\nu_1\in\{-1,1\}}\cdots\sum_{\nu_N\in\{-1,1\}}\exp(\beta[\sum_{i=1}^N u(\nu_i) + J\nu_i\bar{m}_i^e])}. \tag{70}$$

This probability structure is equivalent to the so-called mean field version of the Curie-Weiss model of statistical mechanics, see e.g. Brock and Durlauf (2001b) for further discussion.

For the binary choice model the private utility function can be replaced by a linear private utility function $\tilde{u}(\omega_i) = h\omega_i + k$, with h and k chosen such that $h + k = u(1)$ and $-h + k = u(-1)$. This linearization is possible since the linear functions coincide with the original private utility function on the support of the binary choices (but this trick does not work when more than two choices are possible). Notice that the parameter $h = (u(1) - u(-1))/2$, i.e. h is proportional to the difference in private utility between the two alternatives. Using this linearization and reintroducing expectations of individual choices in (64), the expected value of individual choice ω_i is given by

$$\begin{aligned}
\mathrm{E}[\omega_i] &= 1 \cdot \frac{\exp(\beta h + \beta J(N-1)^{-1}\sum_{j\neq i} m^e_{i,j})}{\exp(\beta h + \beta J(N-1)^{-1}\sum_{j\neq i} m^e_{i,j}) + \exp(-\beta h - \beta J(N-1)^{-1}\sum_{j\neq i} m^e_{i,j})} \\
&\quad (-1) \cdot \frac{\exp(-\beta h - \beta J(N-1)^{-1}\sum_{j\neq i} m^e_{i,j})}{\exp(\beta h + \beta J(N-1)^{-1}\sum_{j\neq i} m^e_{i,j}) + \exp(-\beta h - \beta J(N-1)^{-1}\sum_{j\neq i} m^e_{i,j})} \\
&= \mathrm{Tanh}(\beta h + \beta J(N-1)^{-1}\textstyle\sum_{j\neq i} m^e_{i,j}).
\end{aligned} \tag{71}$$

Brock and Durlauf (2001a) now impose a self-consistent equilibrium or rational expectations equilibrium condition $m^e_{i,j} = E[\omega_j]$ for all i, j. A *rational expectations* or *self-consistent* equilibrium must satisfy

$$E[\omega_i] = \mathrm{Tanh}(\beta h + \beta J(N-1)^{-1}\sum_{j\neq i} E[\omega_j]). \tag{72}$$

By symmetry it follows that $E[\omega_i] = E[\omega_j]$, for all i, j, hence a self-consistent, rational expectations equilibrium average choice level m^* must satisfy

$$m^* = \mathrm{Tanh}(\beta h + \beta J m^*). \tag{73}$$

Brock and Durlauf (2001a) show that a rational expectations equilibrium always exists and, depending upon the parameters, multiple equilibria may exist. More precisely:

- if $\beta J < 1$, then (73) has a unique solution;
- if $\beta J > 1$ and $h = 0$, then (73) has three solutions: 0, one positive solution m^+ and one negative solution m^-;
- if $\beta J > 1$ and $h \neq 0$, then there exists a threshold H (depending on βJ) such that
  - for $|\beta h| < H$, (73) has three solutions, one of which has the same sign as h, and the others possessing opposite signs;
  - for $|\beta h| > H$, (73) has a unique solution with the same sign as h.

Notice that the possibility of multiple equilibria depends on the intensity of choice β, the strength of social interactions J and the difference h in private utility between the two choices. In particular, for each β and J when the difference h is large enough, the equilibrium is unique. Multiplicity of equilibria is most likely when the difference in private utility among alternatives is small and the choice intensity and/or social interaction are strong.

Brock and Durlauf (2001a) also briefly discuss dynamic stability of the steady states of expected choice levels under the assumption of myopic expectations, that is, agents use last period's choice level m_{t-1} as their expectation of others' individual choices. In that case, the dynamic version of (73) becomes

$$m_t = \text{Tanh}(\beta h + \beta J m_{t-1}). \tag{74}$$

Since $f(m) = \text{Tanh}(\beta h + \beta J m)$ is an increasing function of m, it follows easily that (i) if (74) has a unique steady state, then it is globally stable, and (ii) if (74) has three steady states, then the middle one is locally unstable, whereas the smallest and largest steady states both are locally stable. In the case of multiple steady states, the system thus settles down in one of its extremes, where a vast majority of individuals choose one strategy or the other. A large social multiplier exists in such circumstances, that is, small differences in individual utility may lead to large changes at the aggregate level.

6 Heterogeneity and some stylized facts

An important motivation for HAMs has been to explain the stylized facts observed in financial market data. An immediate advantage of a HAM compared to a representative rational agent model is that heterogeneity easily generates large trading volume consistent with empirical observations. Other important stylized facts of financial time series at the daily frequency that have motivated much work on HAMs are: (i) asset prices follow a near unit root process, (ii) asset returns are unpredictable with almost no autocorrelations, (iii) the returns distribution has fat tails, and (iv) financial returns exhibit long range volatility clustering, i.e. slow decay of autocorrelations of squared returns and absolute returns. Facts (i) and (ii) are consistent with a random walk model with a representative rational agent. However, for example, Cutler, Poterba and Summers (1989) have shown that a substantial fraction of stock market fluctuations can *not* be explained by macroeconomic news and that large moves in stock prices are difficult to link with news about major economic or other events. Therefore, a rational agent model has difficulty in explaining fact (iii). One of the most important empirical stylized facts observed in many financial time series is *clustered volatility*, that is, asset price fluctuations are characterized by phases of high volatility interspersed with phases of low volatility. Mandelbrot (1963) was the first to observe this phenomenon. In time series econometrics the class of (generalized) autoregressive conditional hetereroskedastic (G)ARCH-models, pioneered by Engle (1982), has become very popular to describe volatility clustering. However, since news about economic fundamentals do *not* seem to arrive in clusters of high and low volatility, there is no satisfactory representative rational agent explanation of this phenomenon.

In this section we discuss the HAM introduced in Lux (1995,1998) and Lux and Marchesi (1999,2000), which has been successful in explaining the stylized facts (i)-(iv) simultaneously. In particular, clustered volatility arises through the *interaction* and switching between fundamental and chartist trading strategies. Other HAMs explaining these stylized facts include Brock and LeBaron (1996), Arthur et al. (1997), Youssefmir and Huberman (1997), LeBaron et al. (1999), Farmer and Joshi (2002), Kirman and Teyssière (2002), Hommes (2002), Iori (2002), Giardina and Bouchaud (2003) and Gaunersdorfer and Hommes (2005).

6.1 Socio-economic dynamics of speculative markets

The model of Lux (1995,1998) and Lux and Marchesi (1999,2000) describes an asset market with a fixed number N of speculative traders, divided in two groups, *fundamentalists* and *chartists*. Fundamentalists' trading is based upon the fundamental price: they sell (buy) when the price is above (below) the fundamental value. Chartists or technical analysts pursue a combination of *imitative* and *trend following* strategies. At time t, there are n_t^c technical analysts and n_t^f fundamentalists in the market, $n_t^c + n_t^f = N$. The chartists are subdivided into two subgroups: at time t, n_t^+ of them are *optimistic* (bullish) and n_t^- are *pessimistic* (bearish), $n_t^+ + n_t^- = n_t^c$. The number of fundamentalists and (optimistic and pessimistic) chartists changes over time, but to keep the notation simple, we suppress the time index below. The model contains three elements: (1) chartists switching between optimistic and pessimistic beliefs; (2) traders switching between a chartist and a fundamental trading strategy, and (3) a price adjustment process based upon aggregate excess demand.

Contagion behavior of chartists
Chartists switch between an optimistic and a pessimistic mood, depending upon the majority opinion and upon the prevailing price trend. The first element, the contagion behavior, can be motivated as in Keynes' beauty contest that traders try to forecast 'what average opinion expects average opinion to be'. This element is similar in spirit to Kirman's model of opinion formation and Brock and Durlauf's social interaction effects, as discussed in Section 5. An *opinion index*, representing the average opinion among non-fundamentalist traders, is defined as

$$x = \frac{n^+ - n^-}{n^c}, \qquad x \in [-1, +1]. \tag{75}$$

Obviously, $x = 0$ corresponds to the balanced situation where the number of optimists equals the number of pessimists, whereas $x = +1$ (resp. $x = -1$) corresponds to the extreme case where all chartists are optimists (resp. pessimists). It is also useful to define the proportion of chartist traders as

$$z = \frac{n^c}{N}, \qquad z \in [0, +1]. \tag{76}$$

The probabilities for chartists' switching between pessimistic and optimistic depend upon the opinion index x and the price trend (in continuous time) $\dot{p} = dp/dt$. Let

$$U_1 = \alpha_1 x + \alpha_2 \frac{\dot{p}}{\nu_1}, \qquad \alpha_1, \alpha_2 > 0, \tag{77}$$

where the parameters α_1 and α_2 measure the sensitivity of traders to the opinion index (i.e. the behavior of others) resp. their sensitivity to price changes. The switching probabilities are formalized following the synergetics literature, originally developed in physics for interacting particle systems (e.g. Haken (1983)). The probabilities π^{+-} and π^{-+} that chartists switch from pessimistic to optimistic and vice-versa are given by

$$\pi^{+-}(x) = \nu_1 \frac{n^c}{N} e^{U_1}, \qquad \pi^{-+}(x) = \nu_1 \frac{n^c}{N} e^{-U_1}, \qquad \nu_1 > 0. \tag{78}$$

The parameter ν_1 measures the frequency of this type of transition, while the term n^c/N represents the probability for a chartist to meet a chartist.

Switching between chartists and fundamentalists

Agents can also switch between chartists and fundamentalists strategies. These switches are driven by expected or realized excess profits. For chartists, realized excess profit per unit is given by $(y + dp/dt)/p - r$, where y are (constant) nominal dividends of the asset and r is the average (risk adjusted) real return from other investments. It is assumed that $y/p^f = r$, so that at the steady state fundamental price the return from the asset will equal the average return on other investments.

Fundamentalists believe that the asset price will revert back to its fundamental value p^f, and therefore will buy (sell) the asset when its price is below (above) the fundamental value. Fundamentalists expected excess profit is then given by $s|(p - p^f)/p|$. The parameter $s > 0$ may be interpreted as a discount factor, since these are expected excess profits realized only when the price has returned to its fundamental value. Let

$$U_{2,1} = \alpha_3\left(\frac{y + \dot{p}/\nu_2}{p} - R - s\left|\frac{p - p^f}{p}\right|\right), \tag{79}$$

$$U_{2,2} = \alpha_3\left(R - \frac{y + \dot{p}/\nu_2}{p} - s\left|\frac{p - p^f}{p}\right|\right), \tag{80}$$

where α_3 measures the sensitivity of traders to differences in profits. The probabilities to switch from fundamentalists to optimistic chartist, from optimistic chartist to fundamentalists, from fundamentalist to pessimistic chartist resp. from pessimistic chartist to fundamentalists are given by:

$$\pi^{+f} = \nu_2 \frac{n^+}{N} e^{U_{2,1}}, \qquad \pi^{f+} = \nu_2 \frac{n^f}{N} e^{-U_{2,1}}, \tag{81}$$

$$\pi^{-f} = \nu_2 \frac{n^-}{N} e^{U_{2,2}}, \qquad \pi^{f-} = \nu_2 \frac{n^f}{N} e^{-U_{2,2}}, \tag{82}$$

where $\nu_2 > 0$ is a parameter measuring the frequency of this type of transition. Notice the inclusion of the terms n^f/N, n^+/N, n^-/N in the probabilities (81-82), representing the probabilities for a fundamentalist to meet an optimistic chartist, etc. U_1, $U_{2,1}$ and $U_{2,2}$ in fact play the role of a *fitness measure* determining the switching probabilities, similar to Brock and Hommes (1997, 1998)[7]. There is an asymmetry in the fitness measure for chartists and fundamentalists however, since chartists' switching is driven by *realized* profits, whereas fundamentalists' switching is driven by *expected* arbitrage profits which will not be realized until the price has reversed to its fundamental value. Goodhart (1988) pointed out that this asymmetry may bias traders towards chartist strategies. The asymmetry also reflects 'limits to arbitrage' of fundamentalists.

[7]Obviously, these probabilities need to be restricted to the unit interval $[0, 1]$. Note that if one normalizes the expressions for π^{+-} and π^{-+} in (78), π^{+f} and π^{f+} in (81), resp. π^{-f} and π^{f-} in (82) by dividing by their sum, expressions similar to the discrete choice or logit model probabilities used in Brock and Hommes (1997,1998) are obtained (see Sections 7 and 8).

Price formation
Price changes are determined by a *market maker* according to aggregate excess demand of chartists and fundamentalists (cf. Section 4). A chartist buys (sells) a fixed amount t^c of the asset per period when he is optimistic (pessimistic). Using the opinion index x in (75) and the proportion of chartists z in (76), excess demand by chartists is

$$ED^c = (n^+ - n^-)t^c = xzNt^c \equiv xzT^c, \qquad T^c \equiv Nt^c, \tag{83}$$

where T^c denotes the maximum trading volume of chartists. Fundamentalists buy (sell) when the asset price is below (above) its fundamental value, and their excess demand is

$$ED^f = n^f\gamma(p^f - p) = (1-z)N\gamma(p^f - p) \equiv (1-z)T^f(p^f - p), \qquad T^f \equiv N\gamma, \tag{84}$$

where $\gamma > 0$ measures the reaction speed of fundamentalists to price deviations from the fundamental and T^f is a measure of the trading volume of fundamentalists.

A market maker adjusts prices according to aggregate excess demand by

$$\frac{dp}{dt} = \beta[ED^c + ED^f] = \beta[xzT^c + (1-z)T^f(p^f - p)], \tag{85}$$

where β denotes the speed of adjustment.

In their numerical simulations, Lux and Marchesi (1999, 2000) use a stochastic process for the market maker price adjustment. The market maker is assumed to adjust the price to the next higher (lower) possible value (one cent say) within the next time increment with a certain probability depending upon aggregate excess demand. It is also assumed that there are some *noise traders* or *liquidity traders* in the market whose asset demand is random, or alternatively excess demand is observed by the market maker with some imprecision, captured by a noise term μ, normally distributed with standard deviation σ_μ. The transition probabilities for an increase or decrease of the price by an amount $\Delta p = \pm 0.01$ are then given by

$$\begin{aligned} \pi^{\uparrow p} = &\quad \min\{\max\{0, \beta(ED + \mu)\}, 1\}, \\ \pi^{\downarrow p} = &\quad \min\{-\min\{0, \beta(ED + \mu)\}, 1\}. \end{aligned} \tag{86}$$

6.2 Dynamical behavior and time series properties

A formal analysis of this kind of stochastic interacting agent system is possible using the so-called *master equation* for the time evolution of the probability distribution in order to derive differential equations describing a first order approximation of the dynamics of the first moment, i.e. the mean, of the stochastic variables. This approach originates from elementary particle systems in physics and has been followed in the synergetics literature (e.g Haken (1983)) and its applications to social science (e.g. Weidlich and Haag (1983)); see also Aoki (1994, 2002) and references therein for a more detailed treatment and (macro)economic applications. For the current stochastic system the set of differential equations has been derived in Lux (1995, 1998)[8]. The change in the opinion index is governed by:[9]

$$\begin{aligned}
\frac{dx}{dt} &= (\frac{dn^+}{dt} - \frac{dn^-}{dt})/n^c - (n/(n^c)^2)\frac{dn^c}{dt} \\
&= z[(1-x)\pi^{+-} - (1+x)\pi^{-+}] + 0.5(1-z)(1-x^2)(\pi^{+f} - \pi^{f+} + \pi^{f-} - \pi^{-f}), \\
&= 2z\nu_1[\text{Tanh}(U_1) - x]\text{Cosh}(U_1) + (1-z)(1-x^2)\nu_2[\text{Sinh}(U_{2,1}) - \text{Sinh}(U_{2,2})],
\end{aligned} \tag{87}$$

while the change of the proportion of chartist is governed by

$$\begin{aligned}
\frac{dz}{dt} &= \frac{dn^c}{dt}/N = 0.5(1-z)z(1+x)(\pi^{+f} - \pi^{f+}) + 0.5(1-z)z(1-x)(\pi^{-f} - \pi^{f-}), \\
&= (1-z)z(1+x)\nu_2\text{Sinh}(U_{2,1}) + (1-z)z(1-x)\nu_2\text{Sinh}(U_{2,2}).
\end{aligned} \tag{88}$$

Equations (85), (87) and (88) constitute a highly nonlinear 3-D system of differential equations. The system has three types of *steady states*:

(i) $x^* = 0$, $p^* = p^f$, with arbitrary $) \leq z \leq 1$,

(ii) $x^* = 0$, $z^* = 1$, with arbitrary p, and

(iii) $z^* = 0$, $p^* = p^f$, with arbitrary $-1 \leq x \leq +1$.

The most important steady states are of type (i), with the price at its fundamental value, a balanced proportion between optimists and pessimists and an arbitrary proportion of chartists; there exists a continuum of steady states of type (i). At type (ii) steady states the market is completely dominated by chartists, with balanced proportion between optimists and pessimists, and an arbitrary price level. Type (iii) steady states correspond to the other extreme where the market is completely dominated by fundamentalists, with the price at its fundamental value. These extreme cases (ii) and (iii) act as absorbing states of the system. In the numerical simulations of Lux and Marchesi (1999, 2000) these absorbing states are avoided by additional borderline conditions.

Concerning the *(in)stability* of steady state type (i) it should first be noted that, since for any $0 \leq z \leq 1$ such a steady state exists, the corresponding Jacobian matrix of the mean value differential equation system has a zero root, or equivalently, the corresponding discrete system has a unit root. For the stochastic system one thus expects that the proportion of chartists z follows a path close to a random walk, especially when the price is close to the fundamental and the proportions of optimists and pessimists are balanced. Lux (1997) and Lux and Marchesi (2000) provide precise (in)stability conditions of steady states of type (i), which can be summarized as follows. When the parameters α_1, α_2 and α_3 measuring traders sensitivity w.r.t. the opinion index, price changes and profits are larger than some critical value, all steady states of type (i) are repelling. When these sensitivity parameters are below their critical value, the (in)stability depends upon the corresponding proportion z^* of chartists; when this proportion z^* exceeds a critical value, the steady state becomes repelling.

[8]Lux (1997) uses the master equation approach to derive an approximate system of differential equations describing the dynamical behavior of the first *two* moments, the mean and the co-variances, of the stochastic variables.

[9]Recall that $\text{Sinh}(y) = (e^y - e^{-y})/2$, $\text{Cosh}(y) = (e^y + e^{-y})/2$ and $\text{Tanh}(y) = \text{Sinh}(y)/\text{Cosh}(y)$.

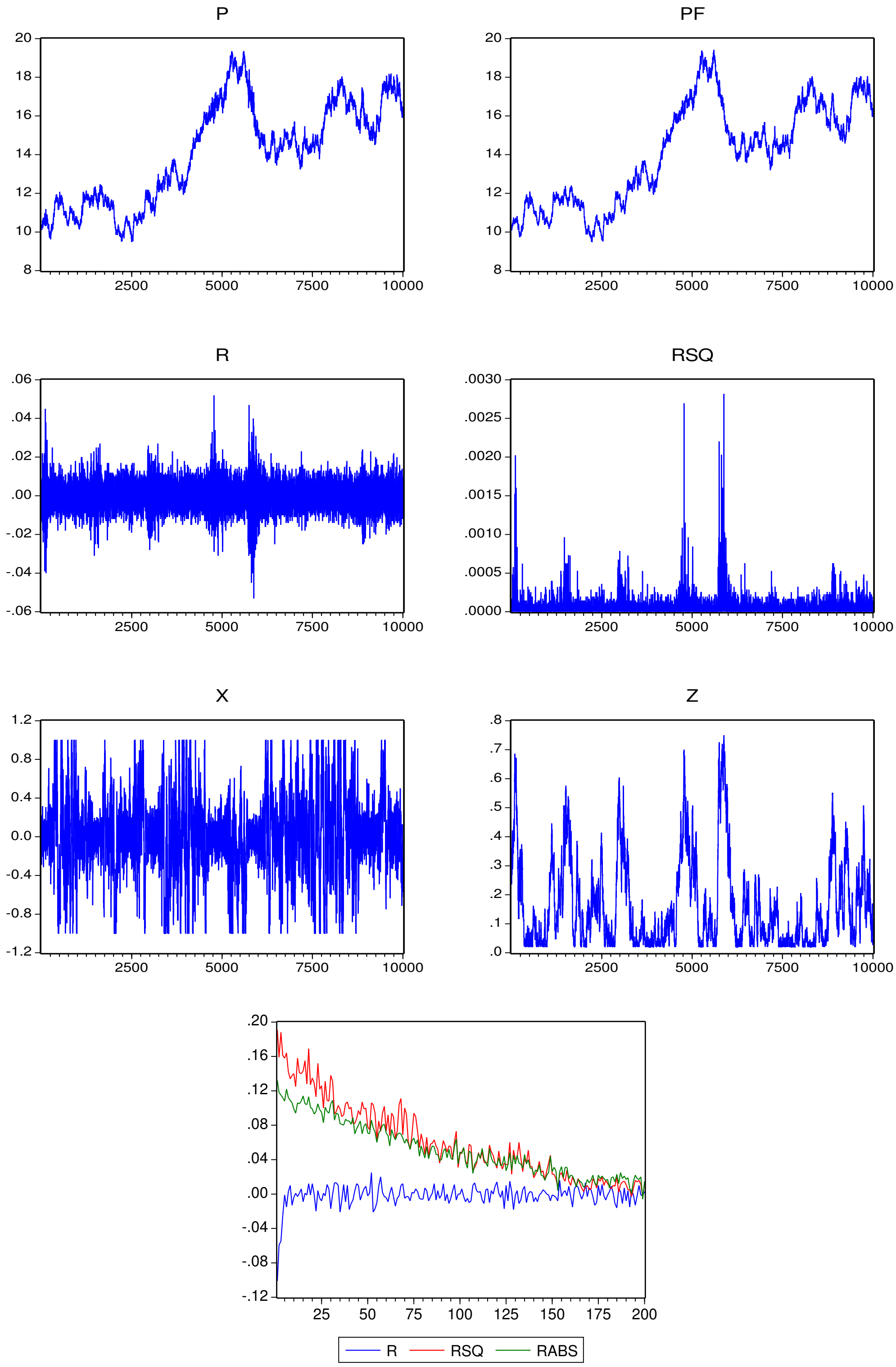

Figure 9: Time series of prices (top left), fundamental price (top right), returns (second panel, left), squared returns (second panel, right), opinion index (third panel, left), fraction of chartists (third panel, right) and autocorrelation patterns (bottom panel) of returns, squared returns and absolute returns. Parameters: $N = 500$, $\nu_1 = 3, \nu_2 = 2, \beta = 6, T_c(\equiv Nt_c) = 10, T_f(\equiv N\gamma) = 5, \alpha_1 = 0.6, \alpha_2 = 0.2, \alpha_1 = 0.5, p_f = 10, y = 0.004$, $R = 0.0004$, $s = 0.75$ and $\sigma_\mu = 0.05$.

Figure 9 shows simulated time series of the model as well as autocorrelations of returns, squared returns and absolute returns[10]. In this simulation, the price stays fairly close to its fundamental value most of the time, because for these parameter values the steady states of type (i) are *not* unstable[11]. Prices follow a near unit root process and financial returns are unpredictable with little autocorrelations (except some small negative autocorrelations at the first lag). Autocorrelations of squared returns and absolute returns are positive and decay slowly, showing long range volatility clustering. The high volatility phase is due to noise amplification through the interactions of agents at the micro-level and coincides with a large proportion of chartists in the market whose opinion is more or less balanced. Returns also exhibit fat tails and Lux and Marchesi (1999,2000) show that the tail of the returns distribution follows a power law.

Lux and Marchesi (1999, 2000) note that these results are fairly robust w.r.t. choices of the parameters. However, Egenter, Lux and Stauffer (1999) show that a puzzling 'finite size effect' occurs, that is volatility clustering tends to disappear when the number of agents N tends to infinity. This finite size effect seems to be due to some law of large numbers. As N becomes large, the random fluctuations in the opinion index become smaller and the population of chartists remains close to being balanced. As a result, the market becomes dominated by fundamentalists and price changes are mainly driven by fundamentals. Nevertheless, this type of HAM matches some important stylized facts remarkably well. In the last 5 years, physicists have done quite a lot of work in finance in particular looking for *scaling laws* in financial market data. The power law decay of the returns distribution and of autocorrelations of squared returns are examples of such a scaling law. For a discussion and overviews of this literature see e.g. Farmer (1999), Mantegna and Stanley (2000), Bouchaud (2001), Cont (2001) and Mandelbrot (2001ab).

[10] I would like to thank Thomas Lux and Timur Yusupov for providing these simulations.

[11] The Jacobian matrix of the steady state has a unit root due to existence of a continuum of steady states. Lux (1997,1998) shows that, for different parameter values the steady state becomes repelling and stable periodic cycles and chaos can occur. In the unstable case, prices persistently deviate from the fundamental value.

7 Costly sophisticated versus cheap simple rules

Herbert Simon (1957) already stressed information gathering and processing costs as an obstacle to fully rational, optimal behavior. Agents must either face search and information gathering costs in using sophisticated, optimal rules or may choose to employ free and easily available simple rules of thumb that perform "reasonably well". In this section we discuss HAMs where agents can choose between costly, sophisticated strategies and simple, but cheap rules of thumb. Simple strategies include naive expectations, adaptive expectations, trend extrapolation, simple technical trading rules, etc. Sophisticated strategies are e.g. fundamental market analysis or predictions of macro economic quantities such as growth, inflation or unemployment, which usually require costly information gathering. In subsection 7.1 we briefly discuss some early, stimulating examples due to Conlisk (1980), Evans and Ramey (1992) and Sethi and Franke (1995), whereas subsection 7.2 discusses the model of Brock and Hommes (1997a) of endogenous selection of costly, sophisticated versus cheap, simple expectations rules.

7.1 Examples

An interesting and early dynamic model with *costly optimizers versus cheap imitators* has been introduced by Conlisk (1980). There are two types of agents, rational optimizers and simple imitators, who try to minimize a quadratic loss function depending upon their choice and an exogenously generated stochastic state of the economy. Optimizers pay a cost for their optimal strategy, to cover the cost of analyzing the decision problem at hand, searching the market or preparing or reading consumer reports, etc. Non-optimizers' behavior is imitative and they adapt their behavior in the direction of last period's observed average optimal choice. Non-optimizers "will make mistakes, but avoid the costs of avoiding mistakes". The mix between the two types evolves over time according to the relative average performance of the two strategies. Conlisk (1980) shows that if the average loss for imitators exceeds the costs for optimizers, imitators can not survive and in the long run optimizers completely dominate the economy. Stated differently, "*Imitation can have no redeeming merit when optimization is cheap enough*" (Conlisk, 1980, p.282). In contrast, if the cost of optimizing is substantial imitators will not disappear but can survive, and both optimizers and imitators will coexist in the long run. Note that in this model the state of the economy evolves according to an *exogenous* stochastic process and is *not* affected by the behavior of the optimizers and imitators.

Evans and Ramey (1992) consider a dynamic macroeconomic model where calculation of rational expectations is costly. Agents have preferences over expectational errors and calculation costs, and in each period choose optimally whether or not to calculate expectations at costs C or keep the same expectation at no costs. The strategy choice is coupled endogenously to the market dynamics. Evans and Ramey discuss the possibility of different types of equilibria. In a *calculation equilibrium* agents start close enough to a REE, it is optimal to never calculate and the system stays close to the REE. Two stage equilibria are characterized by the system starting off far from REE, so that all agents choose to calculate until the point where the system is close enough to REE and it becomes optimal for all agents to switch to 'never calculate'. Sethi and Franke (1995) consider a macroeconomic model with *evolu-*

tionary dynamics and endogenous switching between naive agents using costless adaptive expectations and sophisticated agents using costly rational expectations. Dynamics of output are driven by exogenous shocks to production costs. Strategy fractions are updated according to the relative success of the strategies. Naive agents generally persist in the market, especially when optimization is costly. Both in Evans and Ramey (1992) and Sethi and Franke (1995) market dynamics and strategy selection are endogenously coupled and the state of the economy and the population of strategies co-evolve over time. These examples however are *globally stable*: in the absence of any exogenous random shocks to the economy, both dynamic models converge to a globally stable steady state with all agents using the simple, freely available strategy.

7.2 Rational versus naive expectations

Brock and Hommes (1997a), henceforth BH97a, introduce a model of endogenous, evolutionary selection of heterogeneous expectations rules. In particular, BH97a consider evolutionary switching between a costly sophisticated forecasting strategy, such as rational expectations, versus a free, simple rule of thumb strategy such as naive expectations. They introduce the concept of *Adaptive Rational Equilibrium Dynamics (ARED)*, an endogenous coupling between market equilibrium dynamics and evolutionary selection of expectations rules. The ARED describes evolutionary dynamics among competing prediction strategies, in which the state of the economy and the distribution of agents over different expectation rules co-evolve over time.

Agents can choose between H different (prediction) strategies and update their choice over time. Strategies that have been more successful in the recent past are selected more often than less successful strategies. More precisely, the fraction n_{ht} of traders using strategy h are updated according to an evolutionary *fitness measure* or *performance measure*, such as (a weighted sum of) past realized profits. All fitness measures are publically available (e.g. published in newspapers), but subject to noise e.g. due to measurement error or non-observable characteristics. Fitness of strategy h is given by a random utility model

$$\tilde{U}_{ht} = U_{ht} + \epsilon_{ht}, \tag{89}$$

where U_{ht} is the *deterministic part* of the fitness measure and ϵ_{ht} represents the noise in the observed fitness of strategy h at date t. Assuming that the noise ϵ_{ht} is IID across types and drawn from a double exponential distribution, in the limit as the number of agents goes to infinity, the probability that an agent chooses strategy h is given by the well known *multinomial logit model* or 'Gibbs' probabilities

$$n_{ht} = \frac{\exp(\beta U_{ht})}{Z_t}, \qquad Z_t = \sum_{h=1}^{H} \exp(\beta U_{ht}), \tag{90}$$

where Z_t is a normalization factor for the fractions n_{ht} to add up to 1. Manski and McFadden (1981) and Anderson, de Palma and Thisse (1993) give an extensive overview and discussion of discrete choice models, in particular the multinomial logit model, and their applications in economics. The crucial feature of (90) is that the higher the fitness of trading strategy h, the more agents will select strategy h. The *intensity of choice* parameter $\beta > 0$ in (90) measures how sensitive agents are to selecting the optimal prediction strategy. This intensity of choice β is inversely related to the variance of the noise ϵ_{ht}. The extreme case $\beta = 0$ corresponds to noise with infinite variance, so that differences in fitness cannot be observed and all fractions (90) will be equal to $1/H$. The other extreme case $\beta = +\infty$ corresponds to the case without noise, so that the deterministic part of the fitness is observed perfectly and in each period, *all* agents choose the optimal forecast. An increase in the intensity of choice β represents an increase in the degree of rationality w.r.t. evolutionary selection of strategies[12].

BH97a employ the classical cobweb framework to study a HAM with two prediction strategies, costly rational versus free naive expectations. A related, artifical cobweb economy with genetic algorithms learning is studied by Arifovic (1994). The cobweb model describes fluctuations of equilibrium prices in a market for a non-storable consumption good. The good takes one period to produce, so that producers must form price expectations one period ahead. Applications of the cobweb model mainly concern agricultural markets, such as the classical examples of cycles in hog or corn prices. Supply $S(p_t^e)$ is a function of producer's next period expected price, p_t^e and is derived from expected profit maximization, that is, $S(p_t^e) = \text{argmax}_{q_t} \{p_t^e q_t - c(q_t)\} = (c')^{-1}(p_t^e)$, where $c(\cdot)$ is the cost function. BH97a assume a quadratic cost function $c(q) = q^2/(2s)$, so that the supply curve is linear, $S(p_t^e) = sp_t^e$, $s > 0$. Consumer demand is linearly decreasing in the market price p_t and given by $D(p_t) = a - dp_t$, $d > 0$.[13]

Producers can choose between two different forecasting rules. They can either buy a sophisticated, rational expectations (perfect foresight) forecast at positive per period information cost $C \geq 0$, or freely obtain the simple, naive forecast. The two forecasting rules are thus $p_{1,t}^e = p_t$ and $p_{2,t}^e = p_{t-1}$. Market equilibrium in the cobweb model with rational versus naive expectations and linear demand and supply is given by

$$a - dp_t = n_{1,t-1} s p_{1,t}^e + n_{2,t-1} s p_{2,t} = n_{t-1}^R s p_t + n_{t-1}^N s p_{t-1}, \tag{91}$$

where $n_{1,t-1} = n_{t-1}^R$ and $n_{2,t-1} = n_{t-1}^N$ are the fractions of producers using the rational respectively naive predictor, at the beginning of period t. Notice that producers using RE have perfect foresight, and therefore must have perfect knowledge about the market equilibrium equation (91), including past prices as well as the fractions of both groups. Consequently, rational agents have perfect knowledge about the beliefs of all other agents. The difference C between the per period information costs for rational and naive expectations represents an extra effort cost producers incur over time when acquiring this perfect knowledge. Solving (91) explicitly for the market equilibrium price yields

$$p_t = \frac{a - n_{t-1}^N s p_{t-1}}{d + n_{t-1}^R s}. \tag{92}$$

[12]The probabilities (90) are also used in game theory, in quantal response equilibria introduced by McKelvey and Palfrey (1995), where $\beta = \infty$ corresponds to a Nash equilibrium. Blume (1993) also uses the same type of probabilities in a game theoretic setting and argues that $\beta = \infty$ corresponds to the noise free case where all weight is given to best response(s). Weisbuch et al. (1998) argue that the logit probabilities (90) can be derived as an optimal response in an exploration-exploitation trade off. They derive (90) from maximizing a linear combination of past profit and new information (using entropy as a measure), with β being the weight given to past profit.

[13]Goeree and Hommes (2000) extend the analysis of the cobweb model with rational versus naive expectations to the case of nonlinear (but monotonic) supply and demand.

When all agents have rational expectations, $p_t \equiv p^* = a/(d+s)$, for all $t \geq 1$, that is, the price jumps immediately to its steady state value p^* where demand and supply intersect. When all agents have naive expectations (92) reduces to the linear difference equation $p_t = (a - sp_{t-1})/d$, leading to the familiar up and down price oscillations around the steady state p^*. Price oscillations under naive expectations are stable (unstable) under the familiar 'cobweb theorem' condition $s/d < 1$ ($s/d > 1$).

To complete the model, the fractions of traders using either rational or naive expectations must be specified. As discussed above, these fractions are updated according to a publically available evolutionary fitness measure associated to each predictor. BH97a focus on the case with the most recent realized net profit as the performance measure for predictor selection.[14] For the rational resp. the naive forecasting strategies with linear supply, the realized profits in period t are given by

$$\pi_t^R = p_t\, S(p_t) - c(S(p_t)) = \frac{s}{2} p_t^2, \tag{93}$$

$$\pi_t^N = p_t\, S(p_{t-1}) - c(S(p_{t-1})) = \frac{s}{2} p_{t-1}(2p_t - p_{t-1}). \tag{94}$$

Notice that the *net* realized profit for rational expectations is given by $\pi_t^R - C$, where C is the per period information cost that has to be paid for obtaining the perfect forecast. The fractions of the two groups are determined by the Logit discrete choice model probabilities, as discussed above. The fraction of agents using the rational expectations predictor in period t equals

$$n_t^R = \frac{\exp(\beta(\pi_t^R - C))}{\exp(\beta(\pi_t^R - C)) + \exp(\beta\, \pi_t^N)}, \tag{95}$$

and the fraction of agents choosing the naive predictor in period t is

$$n_t^N = 1 - n_t^R. \tag{96}$$

A key feature of this evolutionary predictor selection is that agents are boundedly rational, in the sense that most agents use the predictor that has the highest fitness. From (95-96) we have that $n_t^R > n_t^N$ whenever $\pi_t^R - C > \pi_t^N$, although the optimal predictor is not chosen with probability one. The *intensity of choice*, i.e. the parameter β, measures how fast producers switch between the two prediction strategies. For $\beta = 0$, both fractions are fixed over time and equal to $1/2$. In the other extreme case $\beta = \infty$ (the *neoclassical limit*) *all* producers choose the optimal predictor in each period.

The timing of predictor selection in (95) is important. In (92) the old fractions n_{t-1}^R and n_{t-1}^N determine the new equilibrium price p_t. This new equilibrium price p_t is used in the fitness measures (93) and (94) for predictor choice and the new fractions n_t^R and n_t^N are updated according to (95) and (96). These new fractions in turn determine the next equilibrium price p_{t+1}, etc. Equilibrium prices and fractions thus co-evolve over time. BH97a called the coupling between the equilibrium price dynamics and adaptive predictor selection an *Adaptive Rational Equilibrium Dynamics (ARED)* model.

[14] The case where the performance measure is realized net profit of the most recent past period, leads to a two-dimensional dynamical system. The more general case, with a weighted sum of past net realized profits as the fitness measure, leads to higher dimensional systems, which are not as analytically tractable as the two-dimensional case. In this more general higher dimensional case however, numerical simulations suggest similar dynamic behavior.

The model has a unique steady state $(p^*, n^*) = (a/(d+s), 1/(1+\exp(\beta C/2))$, with p^* the price where demand and supply intersect. When there are no costs for rational expectations ($C = 0$), at the steady state the fractions of the two types are exactly balanced. In contrast, for positive information costs for rational expectations ($C > 0$), $n^* < 0.5$, so that at the steady state most agents use the naive forecasting rule. This makes sense, because at the steady state both forecasting rules yield exactly the same forecast, and most agents then prefer the cheap, naive forecast.

If the familiar cobweb stability condition $s/d < 1$ is satisfied, implying that the model is stable under naive expectations, then the heterogeneous cobweb model with rational versus naive expectations has a globally stable steady state, for all β. Prices will then always converge to p^*, and the fraction of rational agents converges to n^*. More interesting dynamics occur when the cobweb model is unstable under naive expectations.

Assume that the market is *unstable under naive expectations*, that is, $s/d > 1$:

1. without information costs ($C = 0$), the steady state is globally stable for all β;
2. with positive information costs ($C > 0$), there is a critical value β_1 such that the steady state is (globally) stable for $0 \le \beta < \beta_1$ and unstable for $\beta > \beta_1$. At $\beta = \beta_1$ a *period doubling bifurcation* occurs and a stable 2-cycle is created;
3. as β increases from 0 to $+\infty$ a *rational route to randomness* occurs, that is, a bifurcation route from a stable steady state to a strange attractor occurs and chaotic price fluctuations arise.

Figure 10 shows an example of a strange attractor, with corresponding chaotic time series of prices p_t and fractions n_t^R of rational producers. Numerical simulations suggest that for (almost) all initial states (p_0, m_0) the orbit converges to this strange attractor. For a high intensity of choice price fluctuations are characterized by an irregular switching between a stable phase, with prices close to the steady state, and an unstable phase with fluctuating prices, as illustrated in Figure 10. There is a strikingly simple *economic intuition* explaining this switching behavior when the intensity of choice is large. Suppose we take an initial state close to the (locally unstable) steady state. Most agents will use the cheap, naive forecasting rule, because it does not pay to buy a costly, sophisticated forecasting rule that yields an almost identical forecast. With most agents using the cheap, naive predictor prices diverge from the steady state, start fluctuating, and net realized profits from the naive predictor decrease. At some point, it becomes profitable to buy the rational expectations forecast, and when the intensity of choice to switch predictors is high, most agents will then switch to rational expectations. As a result, prices are driven back close to the steady state, and the story repeats. Irregular, chaotic price fluctuations thus result from a (boundedly) rational choice between cheap 'free riding' and costly sophisticated prediction[15].

Price fluctuations in this simple evolutionary system are thus characterized by an irregular switching between a low volatility phase with prices close to the fundamental steady state and a high volatility phase with large amplitude price fluctuations. The evolutionary system

[15]Brock and Hommes (1997a) show that for a large intensity of choice, the ARED-cobweb model is close to having a so-called homoclinic orbit, a notion already introduced by Poincaré around 1890, and one of the key features of a chaotic system; see Hommes (2005) for a recent, more detailed discussion.

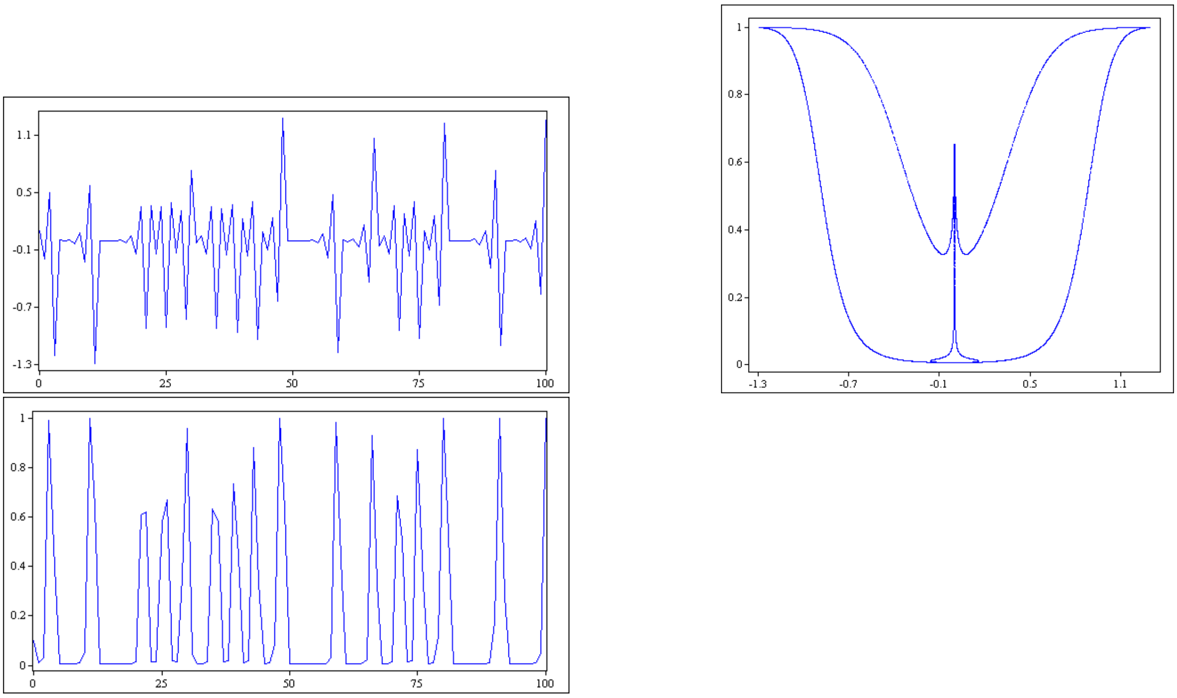

Figure 10: Chaotic time series of price *deviations* from the steady state (top left) and fractions of rational agents (bottom left) and the corresponding strange attractor in the (x, n^R)-phase space (right plot), where $x = p - p^*$ is the deviation from the steady state price. Parameters are: $\beta = 5$, $a = 10$, $d = 0.5$, $s = 1.35$ and $C = 1$.

has a locally destabilizing force due to cheap free riding and a far from the steady state stabilizing force of sophisticated prediction. In this simple evolutionary system, in contrast to the Friedman hypothesis, simple and sophisticated types co-exist in the long run with their fractions fluctuating over time. Due to information gathering costs, rational agents can not drive out naive agents.

Several extensions of the BH97a framework have been considered recently. Branch (2002) investigates the cobweb model with three (rational, adaptive and naive) expectation rules, and Laselle et al. (2005) investigate the case of rational versus adaptive expectations. The same evolutionary framework is applied to an overlapping generations monetary economy by Brock and de Fontnouvelle (2000) and to a Cagan type monetary model by Chiarella and Komin (1999). Branch and McCough (2004) investigate the cobweb model with evolutionary replicator dynamics and obtain similar results; Droste et al. (2002) investigate evolutionary replicator dynamics in a Cournot duopoly model with a Nash rule versus a best reply rule. Branch and Evans (2005) consider a HAM where agents can choose between a number of misspecified econometric models, with a dual learning process of agents learning the model parameters by ordinary least squares (OLS) and strategy fractions updated according to relative performance. De Fontnouvelle (2000) applies the ARED to a financial market model, where agents can choose to buy information about future dividends with high precision, or obtain information with low precision for free. Another recent contribution along these lines is Goldbaum (2005). In the next subsection we discuss an application of the BH97a evolutionary framework to an asset pricing model.

8 Asset pricing model with heterogeneous beliefs

In this section we discuss *Adaptive Belief Systems (ABS)* as introduced by Brock and Hommes (1998), henceforth BH98, a financial market application of the evolutionary selection of expectation rules introduced by Brock and Hommes (1997a). An ABS is in fact a standard discounted value asset pricing model derived from mean-variance maximization, extended to the case of *heterogeneous beliefs*. Agents are boundedly rational and select a forecasting or investment strategy based upon its recent, relative performance. An ABS may be seen as a stylized, to some extent analytically tractable version of more complicated artificial markets, and is in fact similar to the SFI model of Arthur et al. (1997b) and LeBaron et al. (1999) (see also the chapter of LeBaron). A convenient feature of an ABS is that it can be formulated in terms of deviations from a benchmark fundamental and therefore an ABS can be used in experimental and empirical testing of deviations from the RE benchmark.

8.1 The model

Agents can either invest in a risk free asset or in a risky asset. The risk free asset is perfectly elastically supplied and pays a fixed rate of return r; the risky asset (e.g. a stock or a stock market index) pays an uncertain dividend. Let p_t be the price per share (ex-dividend) of the risky asset at time t, and let y_t be the stochastic dividend process of the risky asset. Agents are myopic mean-variance maximizers so that the demand z_{ht} per trader of type h for the risky asset is given by

$$z_{ht} = \frac{E_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1+r)p_t]}{aV_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1+r)p_t]} = \frac{E_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1+r)p_t]}{a\sigma^2}. \tag{97}$$

E_{ht} and V_{ht} denote the 'beliefs' or forecasts of trader type h about conditional expectation and conditional variance of excess return $\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1+r)p_t$ and a is the risk aversion parameter. Bold face variables denote random variables at date $t+1$. For analytical tractability, the conditional variance $V_{ht} = \sigma^2$ is assumed to be equal and constant for all types.[16] Let z^s denote the supply of outside risky shares per investor, assumed to be constant, and let n_{ht} denote the fraction of type h at date t. When there are H different trader types, equilibrium of demand and supply yields

$$\sum_{h=1}^{H} n_{ht} \frac{E_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1+r)p_t]}{a\sigma^2} = z^s. \tag{98}$$

BH98 focus on the special case of zero supply of outside shares, i.e. $z^s = 0$, for which the Walrasian market clearing price satisfies[17]

$$(1+r)p_t = \sum_{h=1}^{H} n_{ht} E_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1}]. \tag{99}$$

[16] Gaunersdorfer (2000) investigates the case with time varying beliefs about variances and Chiarella and He (2000) study heterogeneous risk aversion.

[17] Brock (1997) motivates this special case by introducing a risk adjusted dividend $y^{\#}_{t+1} = y_{t+1} - a\sigma^2 z^s$, and after dropping the superscript "#" obtains the market equilibrium equation (99).

It is well known that in a *homogeneous* world where all agents have rational expectations, the asset price is completely determined by economic fundamentals and given by the discounted sum of expected future dividends:

$$p_t^* = \sum_{k=1}^{\infty} \frac{E_t[\mathbf{y}_{t+k}]}{(1+r)^k}. \tag{100}$$

In general, the properties of the *fundamental price* p_t^* depend upon the stochastic dividend process y_t. In the special case of an IID dividend process y_t, with constant mean $E[y_t] = \bar{y}$, the fundamental price is constant and given by[18]

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y}}{(1+r)^k} = \frac{\bar{y}}{r}. \tag{101}$$

Heterogeneous beliefs

We now discuss traders' expectations about future prices and dividends. As discussed above, beliefs about the conditional variance $V_{ht} = \sigma^2$, for all h, t, are assumed to be equal and constant for all types. Beliefs about future dividends are assumed to be the same for all trader types and equal to the true conditional expectation, that is, $E_{ht}[\mathbf{y}_{t+1}] = E_t[\mathbf{y}_{t+1}]$, for all h, t; in the special case of IID dividends this simplifies to $E_{ht}[\mathbf{y}_{t+1}] = \bar{y}$. All traders are thus able to derive the fundamental price p_t^* in (100) that would prevail in a perfectly rational world. Traders nevertheless believe that in a heterogeneous world prices may *deviate* from their fundamental value p_t^*. It is convenient to introduce the *deviation* from the fundamental price:

$$x_t = p_t - p_t^*, \tag{102}$$

Beliefs about the future price of the risky asset are of the form

$$E_{ht}[\mathbf{p}_{t+1}] = E_t[\mathbf{p}_{t+1}^*] + f_h(x_{t-1}, ..., x_{t-L}), \qquad \text{for all } h, t. \tag{103}$$

Each forecasting rule f_h represents a *model of the market* (e.g. a technical trading rule) according to which type h believes that prices will deviate from the fundamental price. We use the short hand notation $f_{ht} = f_h(x_{t-1}, ..., x_{t-L})$.

An important and convenient consequence of these assumptions concerning traders' beliefs is that the heterogeneous agent market equilibrium equation (99) can be reformulated in deviations from the benchmark fundamental as

$$(1+r)x_t = \sum_{h=1}^{H} n_{ht} E_{ht}[\mathbf{x}_{t+1}] = \sum_{h=1}^{H} n_{ht} f_{ht}. \tag{104}$$

In this general setup, the benchmark rational expectations asset pricing model is *nested* as a special case, with all forecasting strategies $f_h \equiv 0$. In this way, the adaptive belief systems can be used in empirical and experimental testing whether asset prices deviate significantly from a benchmark fundamental.

Evolutionary selection of strategies

The evolutionary part of the model, describing how beliefs are updated over time, follows

[18] Brock and Hommes (1997b), for example, discuss a non-stationary example, where the dividend process is a geometric random walk; see also Hommes (2002).

the endogenous selection of forecasting rules introduced by Brock and Hommes (1997a) as discussed in subsection 7.2. The fractions n_{ht} of trader types are given by the *multinomial logit probabilities* of a discrete choice:

$$n_{ht} = \frac{\exp(\beta U_{h,t-1})}{Z_{t-1}}, \qquad Z_{t-1} = \sum_{h=1}^{H} \exp(\beta U_{h,t-1}). \tag{105}$$

$U_{h,t-1}$ is the fitness measure of strategy h evaluated at the beginning of period t. A natural candidate for evolutionary fitness is (accumulated) *realized profits*, given by

$$U_{ht} = (p_t + y_t - Rp_{t-1})\frac{E_{h,t-1}[\mathbf{p}_t + \mathbf{y}_t - Rp_{t-1}]}{a\sigma^2} + wU_{h,t-1}, \tag{106}$$

where $R = 1 + r$ is the gross risk free rate of return and $0 \leq w \leq 1$ is a *memory* parameter measuring how fast past realized profits are discounted for strategy selection.[19] We will focus on the simplest case with no memory, i.e. $w = 0$, so that fitness U_{ht} equals the most recently observed realized profit. Fitness can now be rewritten in deviations from the fundamental as

$$U_{ht} = (x_t - Rx_{t-1})(\frac{f_{h,t-1} - Rx_{t-1}}{a\sigma^2}). \tag{107}$$

8.2 Few-type examples

BH98 have investigated evolutionary competition between *simple linear* forecasting rules with only *one lag*, i.e.[20]

$$f_{ht} = g_h x_{t-1} + b_h, \tag{108}$$

where g_h is a *trend* parameter and b_h a *bias* parameter. It can be argued that, for a forecasting rule to have any impact in real markets, it has to be simple. For a complicated forecasting rule it seems unlikely that enough traders will coordinate on that particular rule so that it affects market equilibrium prices. Notice that for $g_h = b_h = 0$ the linear forecasting rule (108) reduces to the forecast of *fundamentalists*, i.e. $f_{ht} \equiv 0$, believing that the market price will be equal to the fundamental price p^*, or equivalently that the deviation x from the fundamental will be 0. Notice also that the forecasting rule (108) uses x_{t-1} (or p_{t-1}) as the most recently observed deviation (or price) to forecast x_{t+1} (or p_{t+1}), because the market equilibrium equation (98) has not revealed the equilibrium price p_t yet when forecasts for p_{t+1} are formed. A convenient feature of this setup is that the market equilibrium price p_t is always uniquely defined at all dates t.

This section presents two simple examples of ABS, an example with three and an example with four competing *linear* forecasting rules (108). The ABS becomes (in deviations from

[19] We focus on the case where there are no differences in the costs for the strategies.

[20] Brock and Hommes (1998, pp.1246-1248) also discuss a 2-type example with a costly rational expectations or *perfect foresight* forecasting rule $f_{ht} = x_{t+1}$ versus pure trend followers, and show that the fundamental steady state may become unstable and multiple, non-fundamental steady states may arise. Global dynamics in such an example is difficult to handle, because the system is only *implicitly defined*. Such implicitly defined evolutionary systems cannot be solved explicitely and often they are not even well-defined. See also Arthur (1995) and Hommes (2001) for a discussion of a fully rational agent type within a heterogeneous agents setting.

the fundamental):

$$(1+r)x_t = \sum_{h=1}^{H} n_{ht}(g_h x_{t-1} + b_h) + \epsilon_t \tag{109}$$

$$n_{h,t} = \frac{\exp(\beta U_{h,t-1})}{\sum_{h=1}^{H} \exp(\beta U_{h,t-1})} \tag{110}$$

$$U_{h,t-1} = (x_{t-1} - Rx_{t-2})(\frac{g_h x_{t-3} + b_h - Rx_{t-2}}{a\sigma^2}), \tag{111}$$

where ϵ_t is a small noise term representing uncertainty about economic fundamentals, e.g. random outside supply of the risky asset. The timing of the coupling between the market equilibrium equation (109) and the evolutionary selection of strategies (110) is important. The market equilibrium price p_t (or deviation x_t from the fundamental) in (109) depends upon the fractions n_{ht}. The notation in (110) stresses the fact that these fractions n_{ht} depend upon *past* fitnesses $U_{h,t-1}$, which in turn depend upon past prices p_{t-1} (or deviations x_{t-1}) in periods $t-1$ and further in the past. After the equilibrium price p_t (or the deviation x_t) has been revealed by the market, it will be used in evolutionary updating of beliefs and determining the new fractions $n_{h,t+1}$. These new fractions $n_{h,t+1}$ will then determine a new equilibrium price p_{t+1} (or deviation x_{t+1}), etc. In the ABS, market equilibrium prices and fractions of different trading strategies thus co-evolve over time.

Fundamentalists versus opposite biases

The first example of an ABS has *three* trader types, fundamentalists and two purely *biased* belief, optimists and pessimists expecting a constant price above or below the fundamental price:

$$f_{1t} = 0 \qquad \text{fundamentalists} \tag{112}$$

$$f_{2t} = b \qquad b > 0, \qquad \text{positive bias (optimists)} \tag{113}$$

$$f_{3t} = -b \qquad -b < 0, \qquad \text{negative bias (pessimists).} \tag{114}$$

For low values of the intensity of choice β, the 3-type evolutionary system is stable and the asset price converges to its fundamental value. However, as the intensity of choice increases the fundamental steady becomes unstable due to a *Hopf* bifurcation and the dynamics of the ABS is characterized by cycles around the unstable steady state. This example shows that, even when there are *no* information costs for fundamentalists, they cannot drive out other trader types with opposite biased beliefs. In the evolutionary ABS with high intensity of choice, fundamentalists and biased traders co-exist with their fractions varying over time and asset prices fluctuating around the unstable fundamental steady state. Moreover, Brock and Hommes (1998, p.1259, lemma 9) show that as the intensity of choice tends to infinity the ABS converges to a (globally) stable cycle of period 4. Average profits along this 4-cycle are equal for all three trader types. Hence, if the initial wealth is equal for all three types, then in this evolutionary system in the long run accumulated wealth will be equal for all three types. This example suggests that the Friedman argument that smart fundamental traders will drive out simple habitual rules of speculative traders is not true in general.

Fundamentalists versus trend and bias

The second example of an ABS is an example with *four* trader types, with linear forecasting rules (108) with parameters $g_1 = 0$, $b_1 = 0$; $g_2 = 0.9$, $b_2 = 0.2$; $g_3 = 0.9$, $b_3 = -0.2$ and

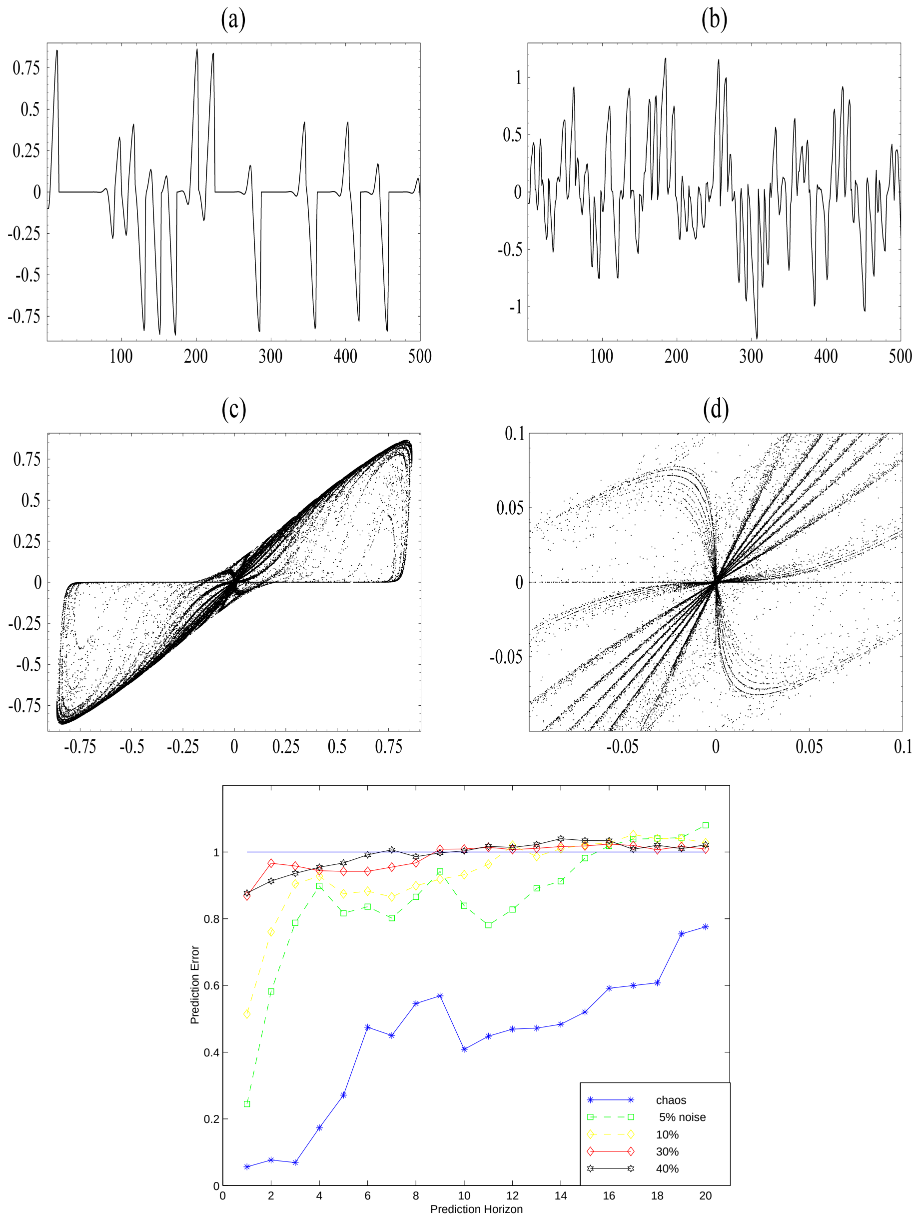

Figure 11: Chaotic (top left) and noisy chaotic (top right) time series of asset prices (deviations from fundamental value) in ABS with four trader types. Strange attractor (middle left) and enlargement of strange attractor (middle right). Belief parameters are: $g_1 = 0$, $b_1 = 0$; $g_2 = 0.9$, $b_2 = 0.2$; $g_3 = 0.9$, $b_3 = -0.2$ and $g_4 = 1 + r = 1.01$, $b_4 = 0$; other parameters are $r = 0.01$, $\beta = 90.5$ and $w = 0$. The bottom right plot shows forecasting errors for the nearest neighbor method applied to noisy chaotic returns series, for different noise levels (see the text). All returns series have close to zero autocorrelations at all lags. The benchmark case of prediction by the mean 0 is represented by the horizontal line at the normalized prediction error 1. Nearest neighbor forecasting applied to the purely deterministic chaotic series leads to much smaller forecasting errors (lowest graph). A noise level of say 10% means that the ratio of the variance of the noise term ϵ_t in (109) and the variance of the deterministic price series is $1/10$. As the noise level slowly increases, the graphs are shifted upwards. Small dynamic noise thus quickly deteriorates forecasting performance.

$g_4 = 1 + r = 1.01$, $b_4 = 0$. The first type are fundamentalists again and the other three types follow a simple linear forecasting rule with one lag. The dynamical behavior is illustrated in Figure 11. For low values of the intensity of choice, the 4-type ABS is stable and the asset price converges to its fundamental value. As the intensity of choice increases, as in the previous three type example, the fundamental steady becomes unstable due to a *Hopf* bifurcation and a stable invariant circle around the unstable fundamental steady state arises, with periodic or quasi-periodic fluctuations. As the intensity of choice further increases, the invariant circle breaks into a strange attractor with chaotic fluctuations. In the evolutionary ABS fundamentalists and chartists co-exist with fractions varying over time and prices moving chaotically around the unstable fundamental steady state.

This 4-type example shows that when traders are driven by short run profits, even when there are *no* information costs, fundamentalists cannot drive out other simple trend following strategies and fail to stabilize price fluctuations towards its fundamental value. As in the three type case, the opposite biases create cyclic behavior, but apparently trend following strategies turn these cycles into unpredictable chaotic fluctuations.

The (noisy) chaotic price fluctuations are characterized by irregular switching between phases of close-to-the-EMH-fundamental-price fluctuations, phases of 'optimism' with prices following an upward trend, and phases of 'pessimism', with (small) sudden market crashes, as illustrated in Figure 11. In fact, in the ABS prices are characterized by an evolutionary switching between the fundamental value and temporary speculative bubbles. In the purely deterministic chaotic case, the start and the direction of the temporary bubbles seem hard to predict. However, once a bubble has started, in the deterministic case, the burst of the bubble seems to be predictable in most of the cases. In the presence of small noise however, as illustrated in Figure 11 (top right), the start, the direction as well as the time of burst of the bubble all seem hard to predict.

In the deterministic chaotic as well as the noisy chaotic case, the autocorrelations of returns are close to zero, so there is little linear predictability in this model. In order to investigate the (un)predictability of this market model in more detail, we employ a so called *nearest neighbor forecasting method* to predict the returns, at lags 1 to 20, for the purely chaotic as well as for several noisy chaotic time series, as illustrated in Figure 11[21]. Nearest neighbor forecasting looks for past patterns close to the most recent pattern, and then yields as the prediction the average value following all nearby past patterns. It follows essentially from Takens' embedding theorem that this method yields good forecasts for deterministic chaotic systems. Figure 11 shows that as the noise level increases, the forecasting performance of the nearest neighbor method quickly deteriorates. Hence, in our simple nonlinear evolutionary ABS with noise it is hard to make good forecasts of future returns. The market is close to being efficient in the sense that there is "no easy free lunch". However, the market is inefficient in the sense that prices exhibit persistent deviations from fundamental value, due to self-fulfilling temporary speculative bubbles driven by short run profit opportunities.

Recently several modifications of ABS have been studied. In BH98a the demand for the risky asset is derived from a constant absolute risk aversion (CARA) utility function. Chiarella and He (2001) consider the case with constant relative risk aversion (CRRA) utility, so that investors' relative wealth affects asset demand and realized asset price, and study wealth and asset price dynamics in such a heterogeneous agents framework[22]. Anufriev and Bottazzi (2005) characterize the type of equilibria and their stability in a HAM with CRRA utility

[21] See e.g. Kantz and Schreiber (1997) for an extensive treatment of nonlinear time series analysis and forecasting techniques such as nearest neighbors. I would like to thank Sebastiano Manzan for providing the nearest neighbor forecasting plot.

[22] In the artifical market of Levy et al. (1994), asset demand is also derived from CRRA utility.

and an arbitrary number of agents. Chiarella, Dieci and Gardini (2002,2005) use CRRA utility in an ABS with a market maker price setting rule. Hommes, Huang and Wang (2005) investigate an ABS with a market maker price setting rule, and find similar dynamical behavior as in the case of a Walrasian market clearing price. Chang (2005) studies the effects of social interactions in an ABS with a Walrasian market clearing price. DeGrauwe and Grimaldi (2005ab) recently applied the ABS framework to exchange rate modeling. A related stochastic model with heterogeneous agents and endogenous strategy switching similar to the ABS has recently been introduced in Föllmer et al. (2005). Scheinkman and Xiong (2004) review related stochastic financial models with heterogeneous beliefs and short sale constraints.

8.3 Many trader types

In most HAMs discussed in this chapter the number of trader types is small, restricted to two, three or four. Analytical tractability can only be obtained at the cost of restriction to just a few types. Brock, Hommes and Wagener (2005), henceforth BHW05, have recently developed a theoretical framework to study evolutionary markets with *many* different trader types. They introduce the notion of *Large Type Limit (LTL)*, a simple, low dimensional approximation of an evolutionary market with many trader types. BHW05 develop the notion of LTL within a fairly general market clearing setting, but here we focus on its application to the asset pricing model.

Recall that in the asset market with H different trader types, the equilibrium price (104), in deviations x_t from the fundamental benchmark, is given by

$$x_t = \frac{1}{1+r} \sum_{h=1}^{H} n_{ht} f_{ht}. \tag{115}$$

Using the *multinomial logit* probabilities (105) for the fractions n_{ht} we get

$$x_t = \frac{1}{1+r} \frac{\sum_{h=1}^{H} \mathrm{e}^{\beta U_{h,t-1}} f_{ht}}{\sum_{h=1}^{H} \mathrm{e}^{\beta U_{h,t-1}}}. \tag{116}$$

The equilibrium equation (116) determines the evolution of the *system with H trader types* - this information is coded in the *evolution map* $\phi_H(\mathbf{x}, \lambda, \theta)$:

$$\phi_H(\mathbf{x}, \lambda, \theta) = \frac{1}{1+r} \frac{\sum_{h=1}^{H} \mathrm{e}^{\beta U(\mathbf{x}, \lambda, \theta_h)} f(\mathbf{x}, \lambda, \theta_h)}{\sum_{h=1}^{H} \mathrm{e}^{\beta U(\mathbf{x}, \lambda, \theta_h)}}, \tag{117}$$

where $\mathbf{x} = (x_{t-1}, x_{t-2}, \cdots)$ is a vector of lagged deviations from the fundamental, λ is a structural parameter vector (e.g. the risk free interest rate r, the risk aversion parameter a, the intensity of choice β, etc.) and the *belief variable* θ_h is now a multidimensional *stochastic* variable which characterizes belief h. At the beginning of the market, a large number H of beliefs is sampled from a general distribution of beliefs. For example, all forecasting rules may be drawn from a linear class with L lags,

$$f_t(\theta_0) = \theta_{00} + \theta_{01} x_{t-1} + \theta_{02} x_{t-2} + \cdots + \theta_{0L} x_{t-L}, \tag{118}$$

with θ_{0h}, $h = 0, \cdots, L$, drawn from a multivariate normal distribution.

The evolution map ϕ_H in (117) determines the dynamical system corresponding to an *asset market with H different belief types*. When the number of trader types H is large, this dynamical system contains a large number of stochastic variables $\theta = (\theta_1, ..., \theta_H)$, where the θ_h are IID, with *distribution function F_μ*. The distribution function of the stochastic belief variable θ_h depends on a multi-dimensional parameter μ, called the *belief parameter*. This setup allows to vary the population out of which the individual beliefs are sampled at the beginning of the market.

Observe that both the denominator and the numerator of the evolution map ϕ_H in (117) may be divided by the number of trader types H and thus may be seen as sample means. The evolution map ψ of the large type limit is then simply obtained by *replacing sample means in the evolution map ϕ_H by population means*:

$$\psi(\mathbf{x}, \lambda, \mu) = \frac{1}{1+r} \frac{\mathrm{E}_\mu \left[\mathrm{e}^{\beta U(\mathbf{x}, \lambda, \theta_0)} f(\mathbf{x}, \lambda, \theta_0) \right]}{\mathrm{E}_\mu \left[\mathrm{e}^{\beta U(\mathbf{x}, \lambda, \theta_0)} \right]} = \frac{1}{1+r} \frac{\int \mathrm{e}^{\beta U(\mathbf{x}, \lambda, \theta_0)} f(\mathbf{x}, \lambda, \theta_0) d\nu_\mu}{\int \mathrm{e}^{\beta U(\mathbf{x}, \lambda, \theta_0)} d\nu_\mu}. \quad (119)$$

Here θ_0 is a stochastic variable which is distributed in the same way as the θ_h, with distribution function F_μ. The *structural* parameter vectors λ of the evolution map ϕ_H and of the LTL evolution map ψ coincide. However, whereas the evolution map ϕ_H in (117) of the heterogeneous agent system contains H randomly drawn multi-dimensional stochastic variables θ_h, the LTL evolution map ψ in (119) only contains the *belief parameter* vector μ describing the joint probability distribution. Taking a large type limit thus leads to a huge reduction in stochastic belief variables.

BHW05 prove an LTL-theorem, saying that, as the number H of trader types tends to infinity, the H-type evolution map ϕ converges almost surely to the LTL-map ψ. The LTL theorem implies that the corresponding LTL dynamical system is a good approximation of the dynamical behavior in a heterogeneous asset market when the number of belief types H is large. In particular, all *generic* and *persistent* dynamic properties will be preserved with high probability. For example, if the LTL-map exhibits a bifurcation route to chaos for one of the structural parameters, then, if the number of trader types H is large, the H-type system also exhibits such a bifurcation route to chaos with high probability.

For example, in the case of linear forecasting rules (118) with three lags ($L = 3$), the corresponding LTL becomes a 5-D nonlinear system given by

$$\begin{aligned} (1+r)x_t &= \mu_0 + \mu_1 x_{t-1} + \mu_2 x_{t-2} + \mu_3 x_{t-3} \\ &\quad + \eta(x_{t-1} - R x_{t-2} + a\sigma^2 z^s)(\sigma_0^2 + \sigma_1^2 x_{t-1} x_{t-3} + \sigma_2^2 x_{t-2} x_{t-4} + \sigma_3^2 x_{t-3} x_{t-5}), \end{aligned} \quad (120)$$

where $\eta = \beta/(a\sigma^2)$ as before. BHW05 show that a bifurcation route to chaos, with asset prices fluctuating around the unstable fundamental steady state, occurs when η increases. This shows that a *rational route to randomness* can occur in an asset market with many different trader types, when traders become increasingly sensitive to differences in fitness (i.e. an increase in the intensity of choice β) or traders become less risk averse (i.e. a decrease of the coefficient of risk aversion a). In a many trader types evolutionary world fundamentalists will in general not drive out all other types and asset prices need not converge to their fundamental value.

Recently Diks and van der Weide (2003,2005) have generalized the notion of LTL and introduced so-called *Continuous Belief Systems* (*CBS*), where the beliefs of traders are distributed according to a continuous density function. The beliefs distribution function and the equilibrium prices co-evolve over time. Assuming a suitable performance measure, e.g. quadratic in the belief parameter θ, the evolution of the distribution of beliefs is determined by the evolution of the first two moments, and analytical expressions for the change of the mean and the variance over time can be obtained. The LTL theory discussed here as well as its extensions can be used to form a bridge between an analytical approach and the literature on evolutionary artificial market simulation models reviewed in the chapter of LeBaron (2005).

9 Concluding remarks and future perspective

The work on heterogeneous agent modeling within the new paradigm of behavioral economics, behavioral finance and bounded rationality is rapidly expanding. This chapter has reviewed HAMs emphasizing models that, at least to some extent, are analytically tractable. The development and analysis of these models requires a combination of analytical and computational tools. The review shows a development from very simple, early models in the seventies and the eighties based on somewhat ad hoc assumptions (e.g. ad hoc demand or supply functions, fixed fractions using the different strategies) to more sophisticated models in the nineties based on micro foundations (e.g. local interactions, social utility, asset demand derived from myopic mean-variance maximization) with switching between different strategies according to an evolutionary fitness measure based upon recent realized performance and social interaction effects. Markets are viewed as *complex adaptive systems*, with the evolutionary selection of expectations rules or trading strategies endogeneously coupled to the market (dis-)equilibrium dynamics. Prices, volume and the population of beliefs and strategies co-evolve over time. In this behavioral world the "wilderness of bounded rationality" is disciplined by parsimony and simplicity of strategies and their relative performance as measured by recent profits, forecasting errors and social utility. Aggregation of interactions of individuals at the micro-level may explain structure and stylized facts at the macro-level.

Dynamic HAMs are highly nonlinear systems, generating a wide range of dynamical behaviors, ranging from simple convergence to a stable steady state to very irregular and unpredictable fluctuations which are highly sensitive to noise. Sophisticated traders, such as fundamentalists or rational arbitrageurs typically act as a *stabilizing force*, pushing prices in the directions of the RE fundamental value. Technical traders, such as feedback traders, trend extrapolators and contrarians typically act as a *destabilizing force*, pushing prices away from the fundamental. When the proportion of chartists believing in a trend exceeds some critical value, the price trend becomes reinforced and the belief becomes self-fulfilling causing prices to deviate from fundamentals. Nonlinear interaction between fundamental traders and chartists can lead to deviations from the fundamental price in the short run, when price trends are reinforced due to technical trading, and mean reversion in the long run, when more agents switch back to fundamental strategies when the deviation from fundamental price becomes too large. Asset prices switch irregularly between temporary bull and bear markets, and are very unpredictable and highly sensitive to noise. Fractions of the different trading strategies fluctuate over time and simple technical trading rules can survive evolutionary competition, and on average chartists may earn profits comparable to the profits earned by fundamentalists or value traders. In financial market applications, simple HAMs can mimic important stylized facts, such as persistence in asset prices, unpredictability of returns at daily horizon, mean reversion at long horizons, excess volatility, clustered volatility and fat tails in asset returns. These models also generate high and persistent trading volume in sharp contrast to no trade theorems in RE models. High trading volume is mainly caused by differences in beliefs. Volatility in asset prices is driven by news about economic fundamentals, which is amplified due to the interaction of different trading strategies. Self-fulfilling trend following investment strategies may cause persistent deviations from fundamental values.

Much more work in this area remains to be done and of the many open issues that remain we can only mention a few. We have seen examples of HAMs where non-rational, non-fundamental traders survive competition in the market. Under which conditions is this true? This important question has also been addressed from a theoretical perspective in the recent *evolutionary finance* literature. Blume and Easley (1992,2002) have shown that in a general equilibrium setting, when markets are incomplete, rational agents are not always able to drive out non-rational traders. Sandroni (2000) shows that in a complete market, agents who do not make accurate predictions are driven out of the market by agents who make accurate predictions.[23] Evstigneev et al. (2002), Hens and Schenk-Hoppé (2005) and Amir et al. (2005) investigate market selection of portfolio rules and investment strategies in asset markets. Applying the theory of random dynamical systems they show that in an incomplete market with short lived assets a unique evolutionary stable strategy distributing wealth according to expected relative payoffs accumulates all wealth. It is an open question whether this result holds for infinite lived assets. It is also an open question whether the Brock and Hommes type of instability will survive in a general equilibrium framework with consumption.

Another important issue is how *memory* in the fitness measure affects stability of evolutionary adaptive systems and survival of technical trading. This question is related to heterogeneity in investors' *time horizon*, both their planning and their evaluation horizon. In a computational framework this problem has been addressed by LeBaron (2002), but simple, analytically tractable models are not available yet. Most dynamic HAMs focus on a market with one risk free and one risky asset, and little attention has been paid to multi risky asset markets. Westerhoff (2004) recently considered multi-asset markets, where chartists can switch their investments between different markets for risky assets. The interaction between the different markets causes complex asset price dynamics, with different markets exhibiting co-movements as well as clustered volatility and fat tails of asset returns. In another recent paper, Böhm and Wenzelburger (2005) apply random dynamical systems to investigate the performance of efficient portfolios in a multi-asset market with heterogeneous investors. A final important question concerns futures or derivative markets. In a homogeneous, rational agent world futures markets are stabilizing because agents can hedge risk and thus force prices closer to their fundamental values. But what happens in a heterogeneous world with boundedly rational agents? Are futures markets stabilizing because risk can be hedged, or

[23] There is an important difference between Sandroni (2000) and e.g. the approach of Brock and Hommes (1997a,1998). Sandroni assumes heterogeneity in expectations about future states of the world, generated by an exogenous stochastic process. These beliefs affect asset prices, but do *not* affect realized states of the world, so that agents with correct beliefs have a comparative advantage in realized utility and asset prices converge to RE prices. In contrast, Brock and Hommes assume correct beliefs about dividends for all agents, but heterogeneous beliefs about prices. These beliefs endogenously affect realized prices. For example, optimistic traders may then survive in the market when enough traders share their optimism, causing the asset price to increase above its RE fundamental value and giving optimists a relatively high return on their investment decision.

will boundedly rational agents take larger positions and destabilize markets?

Expectations play a key role in dynamic HAMs. It is remarkable that relatively little work in laboratory experiments and survey data analysis has focused on dynamic selection of expectation strategies. We mention three recent contributions, and emphasize that much more work needs to be done; see also the chapter of Duffy (2005) for an extensive discussion of the relationship between human subject laboratory experiments and agent-based modeling. Branch (2004) uses survey data on inflation expectations of households to estimate a version of the dynamic HAM of Brock and Hommes (1997), with naive expectations, adaptive expectations and a VAR-forecasting rule. The dynamic HAM fits the survey data best (better than the corresponding homogeneous agent models) with time varying proportions of the three expectations types inversely related to each predictor's MSE. Adam (2005) presents an experimental monetary sticky price economy in which output and inflation depend on expected future inflation. Participants are asked to forecast inflation for about 50 periods, and the average expectation determines next period's output and inflation. In the experimental sessions, output and inflation display considerable persistence and regular cyclical patterns. Such behavior emerges because subjects inflation expectations fail to be captured by rational expectations functions, but instead are well described by simple forecast functions using only one period lagged output and inflation as explanatory variables. Hommes et al. (2005) conduct laboratory experiments, where individuals are asked to forecast an asset price for 50 periods, with realized prices determined endogenously in the laboratory by the Brock-Hommes (1998) asset pricing model with feedback from individual forecasts. In this simple stationary environment, in most cases the asset price does *not* converge to its fundamental value. Agents learn to coordinate on a common, simple prediction rule, e.g. a simple linear trend following rule, and asset prices oscillate around the fundamental value exhibiting short run bubbles and long run mean reversion.

Although there are already quite a number of HAMs, only few attempts have been made to estimate a HAM on economic or financial data. An early attempt has been made by Shiller (1984), who presents a HAM with smart money traders, having rational expectations, versus ordinary investors (whose behavior is in fact not modeled at all). Shiller estimates the fraction of smart money investors over the period 1900-1983, and finds considerable fluctuations of the fraction over a range between 0 and 50%. More recently, Baak (1999) and Chavas (2000) estimate HAMs on hog and beef market data, and found evidence for the heterogeneity of expectations. For the beef market Chavas (2000) finds that about 47% of the beef producers behave naively (using only the last price in their forecast), 18% of the beef producers behaves rationally, whereas 35% behaves quasi-rationally (i.e. use a univariate autoregressive time series model of prices in forecasting). Winker and Gilli (2001) and Gilli and Winker (2003) estimate the model of Kirman (1991,1993) (see subsection 5.1) with fundamentalists and chartists, using the daily DM-US$ exchange rates 1991-2000. Their estimated parameter values correspond to a bimodal distribution of agents, and Gilli and Winker (2003, p.310) conclude that "*... the foreign exchange market can be better characterized by switching moods of the investors than by assuming that the mix of fundamentalists and chartists remains rather stable over time*". Westerhoff and Reits (2003) also estimate an HAM with fundamentalists and chartists to exchange rates and find considerable fluctuations of the market impact of fundamentalists. In a recent paper, Boswijk, Hommes and Manzan (2005) use yearly data of the S&P500 index, 1890-2003, to estimate a version of the Brock and Hommes (1998) asset pricing model with two types of strategies and switching of strategies driven by short run profits. Their estimation yields two different regimes, one stable mean-reverting and one unstable trending regime. Fractions of the two types change considerably over time, and especially in the nineties, the fraction of trend followers becomes large, suggesting that the strong rise in stock prices in the nineties has been exaggerated by trend extrapolation driven by short run profits. All these empirical papers suggest that heterogeneity is important in explaining the data, but much more work is needed to investigate the robustness of this empirical finding.

Much of the work on HAMs is computational and theoretically oriented, but little work has been done on policy implications. The most important difference with a representative rational agent framework is perhaps that in a heterogeneous boundedly rational world, asset price fluctuations exhibit excess volatility. If this is indeed the case, it has important policy implications e.g. concerning the debate on whether a Tobin tax on financial transactions is desirable. In an interesting recent paper, Westerhoff and Dieci (2005) use a HAM to investigate the effectiveness of a Tobin tax. Investors can invest in two different speculative asset markets. If a Tobin tax is imposed on one market, it is stabilized while the other market is destabilized; if a tax is imposed on both markets, price fluctuations in both markets decrease. Another example of a policy oriented paper is Westerhof (2004), who investigates the effectiveness of trading brakes in a HAM. Although much more work is needed to be conclusive on these important issues, these are interesting results illustrating how HAMs can be used to investigate policy issues in future work.

The paradigm of agent-based, behavioral economics, behavioral finance and bounded rationality is rapidly expanding. Heterogeneity is likely to play a key role in this approach, and agent-based computational HAMs deserve high priority in future work. Will an analytical approach survive within more computational oriented research in the 21st century? Computational models are becoming increasingly important and have the advantage that many aspects at the micro level and details of the interaction among agents can be modeled and simulated on a computer. But a problem with large computer simulation models is that there are *too many degrees of freedom* and *too many parameters*. For example, in a computational model often there are many places where noise enters the model at the micro-level, which makes it very difficult to assess the main causes of observed stylized facts at the aggregate, macro level. The search for a (large) computational agent-based HAM capturing the stylized facts as closely as possible deserves high priority. But at the same time one would like to find the *simplest* behavioral HAM (e.g. in terms of number of parameters and variables), with a plausible behavioral story at the micro level, that still captures the most important stylized facts observed at the aggregate level. The simplest HAM can then be used to estimate behavioral heterogeneity in laboratory experimental and/or empirical time series data. Simple and parsimonious HAMs can thus help to discipline the wilderness of agent-based modeling.

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